Classification of Queuing Models

Generally Queuing models may be completely specified in the following symbol form:*(a/b/c):(d/e)*where

a = Probability law for the arrival(or inter arrival)time,

b = Probability law according to which the customers are being served.

c = Number of service stations

d = The maximum number allowed in the system(in service and waiting)

e = Queue Discipline

The above notation is called *Kendal’s Notation*.

Single server Poisson Queue model – I

(M/M/1):(∞/FIFO)

(Single server / Infinite Queue)

In this model we assume that arrival follows a Poisson distribution and services follows an exponential distribution. In this model we assume the arrival rate is λ and service rate is μ.

We know that the Birth – death processes equation is

$$P_n'(t) = -(\lambda_n + \mu_n)P_n(t) + \lambda_{n-1}P_{n-1} + P_{n+1}(t)\mu_{n+1}$$

$$P_0'(t) = -\lambda_0 P_0(t) + P_1(t)\mu_1 \qquad (A)$$

Steady state equations

The steady state equations can got by substituting $\lambda_n = \lambda; \mu_n = \mu; \lambda < \mu$ for all in (A)

Hence the steady state equations are given by

$$-(\lambda+\mu)P_n + \lambda P_{n-1} + P_{n+1}\mu = 0 \qquad .....(1)$$

$$-\lambda P_0 + P_1\mu = 0 \qquad .....(2)$$

Values of P_0 and P_n

By using the equations of (1) and (2) we can get values of P_0 and P_n given by

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \qquad .....(3)$$

$$P_0 = 1 - \left(\frac{\lambda}{\mu}\right) \qquad .....(4)$$

Hence

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left\{1 - \frac{\lambda}{\mu}\right\}$$

Remark:

(1)

$$P_0 = 1 - \left(\frac{\lambda}{\mu}\right)$$

denotes the probability of system being idle

(2)The quantity

$$\frac{\lambda}{\mu} = \rho$$

is called the traffic intensity

Average number (Ls) of customers in the system

Let n be the customers in the system.

Then

$$L_s = E(n) = \sum_{n=0}^{\infty} nP_n$$

$$= \sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\}$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\} \sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left\{1-\left(\frac{\lambda}{\mu}\right)\right\} \sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left\{1-\left(\frac{\lambda}{\mu}\right)\right\} \left(\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n}$$

$$= \{1-\rho\}\, \rho \sum_{n=1}^{\infty} n\rho^{n-1} \quad \text{where } \rho = \frac{\lambda}{\mu}$$

$$= \{1-\rho\}\, \rho \left\{1+2\rho+3\rho^{2}+\ldots\ldots\ldots\right\}$$

$$= \{1-\rho\}\,\rho\,\{1-\rho\}^{-2}$$

$$\left[\text{Q}\,(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots\ldots\ldots\ldots \right]$$

$$= \frac{\rho}{\{1-\rho\}} \qquad \ldots\ldots\ldots\ldots(5)$$

$$= \frac{\frac{\lambda}{\mu}}{\left\{1-\frac{\lambda}{\mu}\right\}}$$

$$= \frac{\lambda}{\{\mu-\lambda\}} \qquad \ldots\ldots\ldots\ldots(6)$$

Remark:

1. By the average number of customers L_s in the system, we mean the number of customers in the queue + the person who is getting serviced.

2. To calculate the expected number of customers or average number of customers in the system we can use the formula given by (5) and (6)

3. The average number of customers in the system can also denoted as $E(n)$.

Average number of customers in the Queue(L_q)

Let n be the customers in the system. Then

$$L_q = E(n) = \sum_{n=1}^{\infty}(n-1)P_n$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\}\sum_{n=1}^{\infty}(n-1)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\} \sum_{n=1}^{\infty} (n-1) \left(\frac{\lambda}{\mu}\right)^2 \left(\frac{\lambda}{\mu}\right)^{n-2}$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\} \left(\frac{\lambda}{\mu}\right)^2 \sum_{n=2}^{\infty} (n-1) \left(\frac{\lambda}{\mu}\right)^{n-2}$$

$$= \{1 - \rho\} \rho^2 \sum_{n=2}^{\infty} (n-1) \rho^{n-2} \quad \text{where } \rho = \frac{\lambda}{\mu}$$

$$= \{1 - \rho\} \rho^2 \{1 + 2\rho + 3\rho^2 + \dots\dots\dots\}$$

$$= \{1-\rho\}\, \rho^{2}\, \{1-\rho\}^{-2}$$

$$\left[\mathrm{Q}\, (1-x)^{-2} = 1 + 2x + 3x^{2} + \ldots\ldots\ldots\ldots \right]$$

$$L_q \quad = \frac{\rho}{\{1-\rho\}} \qquad \ldots\ldots\ldots\ldots(7)$$

$$= \frac{\left(\dfrac{\lambda}{\mu}\right)^{2}}{\left\{1-\dfrac{\lambda}{\mu}\right\}}$$

$$L_q = \frac{\lambda^{2}}{\mu\{\mu-\lambda\}} \qquad \ldots\ldots\ldots(8)$$

Remark :

1)By the average number of customers in the queue we mean the number of customers in the queue excluding the person who is getting serviced

2)To calculate the expected number of customers or average number of customers in the queue we can use the formula given (7) or (8).

3)The average number of customers in the system can also denoted $E(N_q)$

The value of L_q can also be calculated by

$$L_q = L_s - \frac{\lambda}{\mu} \qquad \text{.......(9)}$$

Probability that the number of customers in the system exceeds k.

$$P(n > k) = \sum_{n=k+1}^{\infty} P_n \quad = \sum_{n=k+1}^{\infty} \left\{ 1 - \left(\frac{\lambda}{\mu} \right) \right\} \left(\frac{\lambda}{\mu} \right)^n$$

$$\left[\text{Q}\; P_n = \left(1 - \frac{\lambda}{\mu} \right) \left(\frac{\lambda}{\mu} \right)^n \right]$$

$$= \left\{ 1 - \left(\frac{\lambda}{\mu} \right) \right\} \sum_{n=k+1}^{\infty} \left(\frac{\lambda}{\mu} \right)^n$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\}\left[\left(\frac{\lambda}{\mu}\right)^{k+1} + \left(\frac{\lambda}{\mu}\right)^{k+2} + \ldots.\right]$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\}\left(\frac{\lambda}{\mu}\right)^{k+1}\left[1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^{2} + \ldots..\right]$$

$$= \left\{1 - \left(\frac{\lambda}{\mu}\right)\right\}\left(\frac{\lambda}{\mu}\right)^{k+1}\left[1 - \frac{\lambda}{\mu}\right]^{-1}$$

$$= \left(\frac{\lambda}{\mu}\right)^{k+1} \qquad \text{........(10)}$$

$$= \rho^{k+1} \qquad \text{........(11)}$$

Average number L_w of customers in the nonempty queue

Since the queue is nonempty, we have

$$L_w = E\{(N-1)/(N-1)>0\}$$

$$= \frac{E(N-1)}{P\{(N-1)>0\}}$$

$$= \frac{\lambda^2}{\mu\{\mu-\lambda\}} \times \frac{1}{\sum_{n=2}^{\infty} P_n} \qquad \text{(by (8))}$$

$$= \frac{\lambda^2}{\mu\{\mu-\lambda\}} \times \frac{\mu^2}{\lambda^2} \qquad \text{(by (10))}$$

$$= \frac{\mu}{\mu-\lambda} \qquad \text{......(12)}$$

Probability density function of the waiting time in the system.

Let W_s be the continuous random variable that represents the waiting time of a customer in the system.

Note that this waiting includes the waiting time of the customer in queue as well as the service time.

Let its p.d.f be *f(w)* and let *f(w/n)* be the density function of W_s subject to the condition that there are *n* customers in the system when the customer arrives.

$$\therefore\ f(w) = \sum_{n=0}^{\infty} f(w/n)P_n \qquad \text{......(13)}$$

Now $f(w/n)$ =p.d.f of sum of $(n+1)$ service times

=p.d.f of sum of $(n+1)$independent random variables

Since each of the $(n+1)$ independent random variables are exponentially distributed with parameter μ , we have

$$f(w/n) = \frac{\mu^{n+1}}{n!} e^{-\mu w} w^n, w > 0$$

Which is the p.d.f of the erlang distribution .

Hence (13) becomes

$$f(w) = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{n!} e^{-\mu w} w^n P_n$$

$$= \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{n!} e^{-\mu w} w^n \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \mu e^{-\mu w} \sum_{n=0}^{\infty} \frac{(\lambda w)^n}{n!} = (\mu - \lambda) e^{-\mu w} e^{\lambda w}$$

$$= (\mu - \lambda) e^{-(\mu - \lambda) w} \qquad \text{........(14)}$$

which is the p.d.f of the exponential distribution

with parameter μ -λ.

Average waiting time of a customers in the system

Let W_s be the waiting time of a customer in the system. Then W_s follows a exponential distribution with parameter μ -λ

$$W_s = \frac{1}{\mu - \lambda} \qquad \text{......(15)}$$

(since , if λ is the parameter of the exponential distribution then the mean of the Distribution is $1/\lambda$)

Average waiting time of the customer in the queue

Let W_q be the waiting time of a customer in the system. Then

$$W_q = \frac{1}{\mu-\lambda} - \frac{1}{\mu}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)} \qquad \text{......(16)}$$

Probability that the waiting time of a customer in the system exceeds *t*.

$$P(W_s > t) = \int_t^\infty f(w)dw = \int_t^\infty (\mu-\lambda)e^{-(\mu-\lambda)w}dw$$

$$= -\left[e^{-(\mu-\lambda)w}\right]_t^\infty = e^{-(\mu-\lambda)t} \qquad \text{.........(17)}$$

Probability density function of the waiting time in the system

Let W_q be the continuous random variable that represents the waiting time of a customer in the system. Note that this waiting time does not include the waiting time of the customer in service.

Let its p.d.f be $g(w)$ and let $g(w/n)$ be the density function of W_q subject to the condition that there are n customers in the system when the customer arrives.

$$\therefore g(w) = \sum_{n=0}^{\infty} g(w/n)\,p_n \qquad .....(18)$$

Now *g(w/n)* = p.d.f of sum of *n* service times

Hence

$$g(w/n) = \frac{\mu^n}{(n-1)!} e^{-\mu w} w^{n-1}, w > 0$$

Hence (18) becomes

$$\therefore g(w) = \sum_{n=1}^{\infty} \frac{\mu^n}{n-1} e^{-\mu w} w^{n-1} P_n$$

$$= \sum_{n=1}^{\infty} \frac{\mu^n}{n-1} e^{-\mu w} w^{n-1} \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-\mu w} e^{\lambda w}$$

$$= \begin{cases} \frac{\lambda}{\mu}(\mu - \lambda) e^{-(\mu-\lambda)w} & (w > 0) \\ \frac{\lambda}{\mu}(\mu - \lambda) & (w = 0) \end{cases}$$

Little’s formula

We have a relation between W_s, W_q, L_s, L_q given as follows

$$L_s = \frac{\lambda}{\mu - \lambda} = \lambda W_s$$

$$L_s = \frac{\lambda}{\mu - \lambda} = L_q + \frac{\lambda}{\mu}$$

$$W_s = \frac{1}{\mu - \lambda} = W_q + \frac{1}{\mu}$$

$$L_q = \frac{\lambda^2}{\mu - \lambda} = \lambda W_q$$

Remark:

Using Little’s formula, if one value is calculated then the other 3 values can be found using the above relation

Problems

Example 1

Arrival rate of telephone calls at a telephone booth is according to the Poisson distribution with an average time of 9 minutes between two consecutive arrivals. The length of telephone call is assume to be exponentially distributed with mean 3 minutes.

(a) Determine the probability that a person arriving at the booth will have to wait.

(b) Find the average Queue length that forms from time to time

(c)The telephone company will install a second booth when convinced that an arrival would expect to have to wait at least four minutes for the phone. Find the increase in flow of arrivals which will justify a second booth.

(d) What is the probability that an arrival will have to wait for more than 10 minutes before the phone is free?

(e) What is the probability that an arrival will have to wait for more than 10minutes before the phone is available and the call is also complete?

(f) Find the fraction of a day that the phone will be in use.

Solution:

Step 1: Model identification

Since there is only one telephone booth, the number of service channels is one. Also, since any number of customers can enter the booth, the capacity of the system is infinity. Hence this problem comes under the model $(M/M/1):(\infty/FCFS)$.

(Single server , Infinite queue)

Step 2: Given data

$$\text{Arrival rate, } \lambda = \frac{1}{9} \text{ per minute}$$

$$\text{service rate, } \mu = \frac{1}{3} \text{ per minute}$$

Step 3: To find the following

- The probability that a person arriving at the booth will have to wait i.e.
- The average queue length that forms from time to time i.e. L_s

- The increase in flow of arrivals which will justify a second booth(we need to apply a different technique)
- The probability that an arrival will have to wait for more than 10 minutes before the phone is free i.e. $P(W_s > 10)$

- The probability that an arrival will have to wait for more than 10 minutes before the phone is available and the call is also complete i.e. probability (time in system > 10)
- Find the fraction of a day that the phone will be in use. i.e. . ρ.

Step 4 : Required computations

- Probability that a person will have to wait

$$\rho = \frac{\lambda}{\mu} = \frac{1/9}{3/9}$$

$$= 0.33$$

- Average queue length that forms from time to time

$$L_s = \frac{\mu}{\mu - \lambda}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} - \frac{1}{9}} = \frac{\frac{1}{3}}{\frac{2}{9}} = \frac{9}{6}$$

$$= 1.5$$

- Average waiting time in the queue

$$W_q = \frac{\lambda_1}{\mu(\mu - \lambda_1)} \qquad \text{(We use } \lambda_1 \text{ here as we have used } \lambda \text{ already)}$$

$$4 = \frac{\lambda_1}{\frac{1}{3}(\frac{1}{3} - \lambda_1)} \qquad \text{(Q The waiting time to install a second booth is 4 minutes)}$$

$$\frac{4}{9} - \frac{4\lambda_1}{3} = \lambda_1$$

$$\frac{4}{9} = \lambda_1 + \frac{4\lambda_1}{3}$$

$$\frac{4}{9} = \frac{7\lambda_1}{3}$$

$$\lambda_1 = \frac{4}{21} \text{ arrivals / minute}$$

$$\therefore \text{Increase in flow of arrivals} = \frac{4}{21} - \frac{1}{9}$$

$$= \frac{5}{63} \text{ per minute}$$

•Probability of waiting time more than 10 minutes

$= P(W_s > 10)$

$$= \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda) e^{-(\mu-\lambda)t} dt$$

(p.d.f of a exponential distribution)

$$= \frac{\lambda}{\mu} \int_{10}^{\infty} (\mu - \lambda) e^{-(\mu-\lambda)t} dt$$

$$= \frac{\lambda}{\mu} (\mu - \lambda) \left[\frac{e^{-(\mu-\lambda)t}}{-(\mu - \lambda)} \right]_{10}^{\infty}$$

$$= \frac{\lambda}{\mu} \left[0 + e^{-10(\mu-\lambda)} \right]$$

$$= \frac{1}{3} e^{-10\left(\frac{1}{3} - \frac{1}{9}\right)}$$

$$= \frac{1}{3} e^{-\frac{20}{9}}$$

$$= \frac{1}{30} \text{ (approximately)}$$

•Probability (time in system > 10)

$$= \int_{10}^{\infty} (\mu - \lambda) e^{-(\mu-\lambda)t} dt$$

(p.d.f of a exponential distribution)

$$=\frac{\mu}{\lambda}\left\{\frac{\lambda}{\mu}\int_{10}^{\infty}(\mu-\lambda)e^{-(\mu-\lambda)t}dt\right\}$$

$$=\frac{\mu}{\lambda}\left(\frac{1}{30}\right)=\frac{\frac{1}{3}}{\frac{1}{9}}\left(\frac{1}{30}\right) \qquad (By\ (d))$$

$$=0.1$$

- The expected fraction of day that the phone will be in use is equal to

$$= \frac{\lambda}{\mu}$$

$$= \frac{\frac{1}{9}}{\frac{1}{3}}$$

$$= 0.33$$

Example 2:

Customers arrive at a one man barber shop according to a Poisson process with an mean inter-arrival time of 12 minutes. customers spend a average of 10 minutes in the barber's chair.

(a)What is the expected number of customers in the barber shop and in the queue?

(b)Calculate the percent of time an arrival can walk straight into the barber’s chair without having to wait

(c)How much time can a customer expect to spend in the barber’s shop?

(d)Management will provide another chair and hire another barber, when the customer’s waiting time in the shop exceeds 1.25Hr. How much must the average rate of arrivals increase to warrant a second barber?

(e)What is the average time customers spend in the queue?

(f)What is the probability that the waiting time in the system greater than 30 minutes?

(g)Calculate the percentage of customers who have to wait prior to getting into the barber’s chair.

(h)What is the probability that more than 3 customers are in the system?

Step 1: Model Identification

Since there is only one barber, the number of service channels is one. Also, since any number of persons can enter the barber shop, the capacity of the system is infinity. Hence this problem comes under the model

$$\left(M/M/1\right):\left(\infty/FCFS\right).$$

(Single server / infinite Queue)

Step 2: Given data:

$$\text{Mean arrival rate }, \quad \frac{1}{\lambda} = 12$$

$$\therefore \lambda = \frac{1}{12} \quad \text{per minute}$$

$$\text{Mean service rate} = \frac{1}{\mu} = 10$$

$$\therefore \mu = \frac{1}{10} \text{ per minute}$$

Step 3: To find the following:

(a) Expected number of customers in the system i.e., L_s

Expected number of customers in the queue i.e., L_q

(b) P (a customer straightly goes to the barber's chair) i.e., P_0

(c) The time can a customer expect to spend in the barber's shop i.e., W_s

(d) The new arrival rate λ_r (say) , if $W_s > 75$

(e) Average time customers spend in the queue i.e., W_q

(f)Probability that the waiting time in the system is greater than 30 minutes i.e., $P(W>30)$

(g) Percentage of customers who have to wait prior to getting into the barber's chair i.e., $P(W>0)$

(h) The probability that more than 3 customers are in the system

Step 4: Required computations

a) Expected number of customers in the system

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{10} - \frac{1}{12}}$$

$$= 5 \text{ customers}$$

Expected number of customers in queue

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\frac{1}{144}}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{12}\right)}$$

$$= 4.17 \text{ customers}$$

b) P(a customer straightly goes to the barber's chair)

= P(no customer in the system)

(i.e., the system is idle)

$$P_o = 1 - \frac{\lambda}{\mu} = 1 - \frac{\frac{1}{12}}{\frac{1}{10}}$$

$$= \frac{1}{6}$$

$$\therefore \text{Percentage of time on arrival need not wait} = 16.7$$

$$c)\ W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{10} - \frac{1}{12}}$$

$$= 60 \text{ minutes or 1 hour}$$

$$d)\ W_s > 75, \text{ if } \frac{1}{\mu - \lambda_r} > 75, \text{ where } \lambda_r \text{ is the new arrival rate.}$$

$$\text{(i.e) if } \lambda_r > \mu - \frac{1}{75}$$

$$\text{(i.e) if } \lambda_r > \frac{1}{10} - \frac{1}{75}$$

$$\text{(i.e) if } \lambda_r > \frac{13}{150}$$

Hence to warrant a second barber,

the average arrival rate must increase by

$$\frac{13}{150} - \frac{1}{12} = \frac{1}{300} \text{ per minute}$$

$$e)\ W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\frac{1}{12}}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{12}\right)} = 50 \text{ minutes}$$

$$f)\ P(W > t) = e^{-(\mu-\lambda)t}$$

$$P(W > 30) = e^{-\left(\frac{1}{10}-\frac{1}{12}\right)\times 30}$$

$$= e^{-0.5}$$

$$= 0.6065$$

$$g)\ P(a\ customer\ has\ to\ wait) = P(W > 0)$$

$$= 1 - P(W = 0)$$

$$= 1 - P(Number\ of\ customers = 0)$$

$$= 1 - P_o$$

$$= \frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{10}} = \frac{5}{6}$$

$$\therefore \text{Percentage of customers who have to wait} = \frac{5}{6} \times 100 = 83.33$$

$$(h)\ P(N>3) = P_4 + P_5 + P_6 + \ldots..$$

$$= 1 - \{P_0 + P_1 + P_2 + P_3\}$$

$$= 1 - (1 - \frac{\lambda}{\mu})\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3\right)$$

$$\left[\text{Since } P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right), \text{ for } n \geq 0\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{5}{6}\right)^4$$

$$= 0.4823$$

Exercises:

1) In a telephone booth the arrivals are on the average 15 per hour. A call on the average takes 3 minutes. If there are just one phone (Poisson and Exponential arrival), find the expected number of customers in the booth and the idle time of the booth.

[Hint : Model identification – Given λ & μ. Find L_s and P_0

2) A car machine finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. If he repairs the car as how it comes in, arrival rate is Poisson with an average rate of 10 per 8 hour per day

(i) What is the repairman’s expected idle time each day?

(ii) How many jobs are ahead of the average set brought in.

[Model identification, Given λ &μ. Find P_o , L_s]

3) In a given M/M/I queuing system, the average arrivals is 4 customers per minute and $\rho = 0.7$. What is

1) Mean number of customers in the system

2) Mean number of customers in the queue.

3) Probability that the server is idle.

4) Mean waiting time W_s in the system

5) Mean waiting time W_q in the queue

[Model identification – Given λ & ρ. Find μ, L_s, L_q ,P_0, W_s,W_q]

Model 2: (M/M/1) : (N/FIFO)

(Finite capacity, single server

Poisson Queue Model)

Characteristics

The steady state are given by the difference equations

$$-(\lambda_n + \mu_n)P_n + \lambda_{n-1}P_{n-1} + P_{n+1}\mu_{n+1} = 0 \quad (n > 0) \qquad .....(1)$$

and

$$-\lambda_0 P_0 + P_1\mu_1 = 0 \qquad .....(2)$$

This is a typical equations model where the number of customers in the system is limited. In other words, the system can accommodate only finite number of customers in the system. If a customer arrives and the queue is full, the customers leaves the system without joining the queue.

Hence for this model

$$\mu_n = \mu,\ n = 1, 2, 3, \ldots$$

$$\& \lambda_{n =} \lambda,\ \text{for } n = 0, 1, 2, 3, \ldots (n-1) \text{ and is zero for } n = N, N+1, \ldots$$

Using these values in the difference equations given in (1) and (2), we have

$$P_1 \mu = \lambda P_0 \qquad \ldots..(3)$$

and hence

$$P_1 = \frac{\lambda}{\mu} P_0 \qquad \ldots..(4)$$

Also, $P_{n+1}\,\mu = (\lambda+\mu)\,P_n - \lambda\,P_{n-1}\;, 0 < n < N$ .....(5)

and $P_N\,\mu = \lambda\,P_{N-1}\;$, for $n = N$ .....(6)

From (5) $P_2\,\mu = (\lambda+\mu)\,P_1 - \lambda\,P_0$

$$\Rightarrow \quad P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \qquad \textit{and so on} \;.....(7)$$

$$\textit{In general,} \quad P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \;, 0 < n < N \qquad .....(8)$$

$$\textit{From } (5), \qquad P_N = \frac{\lambda}{\mu} P_{N-1} \qquad \text{.....(9)}$$

$$\Rightarrow \qquad P_N = \frac{\lambda}{\mu}\left(\frac{\lambda}{\mu}\right)^{N-1} P_0$$

$$\Rightarrow \qquad P_N = \left(\frac{\lambda}{\mu}\right)^{N} P_0 \qquad \text{.....(10)}$$

Since total probability is one, we have

$$\sum_{n=0}^{N} P_n = 1$$

$$\Rightarrow \quad \sum_{n=0}^{N} \left(\frac{\lambda}{\mu} \right)^{N} P_0 = 1$$

$$\Rightarrow \quad P_0 \sum_{n=0}^{N} \left(\frac{\lambda}{\mu} \right)^{N} = 1$$

$$\Rightarrow P_0 \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu} \right)^{2} + \ldots + \left(\frac{\lambda}{\mu} \right)^{N} \right) = 1$$

$$\Rightarrow \quad P_0 \frac{\left(1 - \left(\frac{\lambda}{\mu} \right)^{N+1} \right)}{1 - \frac{\lambda}{\mu}} = 1 \text{, } valid \;\; for \; \lambda \neq \mu$$

$$\Rightarrow \quad P_0 = \frac{1-\frac{\lambda}{\mu}}{\left(1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right)}, valid\ for\ \lambda \neq \mu$$

$$\text{If } \lambda = \mu, \text{then} \frac{\lambda}{\mu} = 1. \text{Hence } P_N = \left(\frac{\lambda}{\mu}\right)^N P_o = P_o$$

$$\therefore \sum_{n=0}^{N} P_n = 1 \Rightarrow \sum_{n=0}^{N} P_0 = 1$$

$$\therefore (N+1)\, P_0 = 1 \Rightarrow P_0 = \frac{1}{(N+1)}$$

Hence from (10), we have

$$P_n = \left(\frac{\lambda}{\mu}\right)^N \frac{1-\frac{\lambda}{\mu}}{\left(1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right)}, \quad for\ \lambda \neq \mu$$

and $P_n = \dfrac{1}{(N+1)}$ for $\lambda = \mu$

Similar to the first model, we can find the following

- Probability that the system is idle.

$$P_0 = \frac{1-\rho}{1-(\rho)^{N+1}} \qquad \left[\because \rho = \frac{\lambda}{\mu}\right]$$

- Average number of customers in the system

$$L_s = E(n)$$

$$= P_o \times \sum_{n=0}^{N} n \rho^n$$

$$=\frac{\lambda}{\mu-\lambda}-\frac{(N+1)\left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}},\ for\ \lambda\neq\mu$$

$$=\frac{k}{2},\ for\ \lambda=\mu$$

- Average queue length

$$L_q=L_s-\frac{\lambda'}{\mu},\ \lambda'=\mu\left(1-P_0\right)\ \text{the effective arrival rate}$$

- Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda'}$$

- Average number of units in the queue

$$L_q = L_s - (1 - P_0)$$

Example 1:

A one – person barbershop has six chairs to accommodate people waiting for hair cut. Assume customers who arrive when all six chairs are full, leave without entering the barbershop. Customers arrive at the average rate of 3 per hour and spend on average of 15 minutes in the barbershop. Then find the

a) the probability a customer can get directly into the barber chair upon arrival.

b) Expected number of customers waiting for hair cut.

c) Effective arrival rate.

d) The time a customer can expect to spend in the barbershop.

Step 1: Model identification

Since there is only one barber, the number of service channels is one. Also, since only six chairs are available, the capacity of the system is finite. Hence this problem comes under the model (M/M/1) : (N/FCFS)

Step 2 : Given data

Arrival rate , $\lambda = 3$ per hour

$$\text{Service rate }, \mu = \frac{1}{15} \text{ per minute}$$

$$= 4 \text{ per hour}$$

$$N = \text{Capacity of the system}$$

$$= \text{Chairs to accomodate waiting people}$$

$$+ \text{ one chair in service}$$

$$= 6 + 1$$

$$= 7$$

Step 3: To find the following

- The probability a customer can get directly into the barber chair upon arrival i.e., P_0 .
- Expected number of customers waiting for hair cut i.e., L_q
- Effective arrival rate i.e., Service rate x$(1 – P_0)$
- The time a customer can expect to spend in the barbershop i.e., $\mathrm{W_q}$

Step 4 : Required computations

• The customer will directly go in barber chair when the system at the time of his arrival is empty. The probability in this situation is given by P_0.

$$We\ know\ that \quad P_0 = \frac{1-\rho}{1-(\rho)^{N+1}} \quad , \rho = \frac{\lambda}{\mu} = \frac{3}{4}$$

$$= \frac{1-\frac{3}{4}}{1-\left(\frac{3}{4}\right)^{7+1}}$$

$$= 0.2778$$

•Expected number of customers waiting for the hair cut(L_q)

$$L_q = L_s - 1 + P_0$$

$$= \frac{\lambda}{\mu - \lambda} - \frac{(N+1)\left(\frac{\lambda}{\mu}\right)^{N+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{N+1}} - 1 + P_0$$

$$= \frac{3}{4-3} - \frac{(7+1)\left(\frac{3}{4}\right)^{7+1}}{1 - \left(\frac{3}{4}\right)^{7+1}} - 1 + 0.2778$$

$$= 3 - \frac{8\left(\frac{3}{4}\right)^8}{1-\left(\frac{3}{4}\right)^8} - 1 + 0.2778$$

$$= 1.38$$

Effective arrival rate = (service rate)$(1 - P_0)$

$$= \mu\left(1 - \frac{\lambda}{\mu}\right)$$

$$= 4(1 - 0.2778)$$

$$= 2.89 \ per \ hour$$

The time a customer can expect to spend in the system (W_s)

$$W_s = \frac{L_s}{\text{Effective arrival rate}}$$

$$= \frac{L_q + 1 - P_o}{\text{Effective arrival rate}}$$

$$= \frac{1.38 + 1 - 0.2778}{2.89}$$

$$= 43.2 \text{ minutes}$$

Example 2:

Trains arrive at the yard every 15 minutes and the service rate is 33 minutes. If the line capacity at the yard is limited to 4 trains, find

- The probability that the yard is empty.
- The average number of trains in the system.

Solution:

Step 1:Model identification:

Since there is only one yard, the number of service channels is one. Also, since line capacity of the yard is limited to 4 trains, the capacity of the system is finite. Hence this problem comes under the model *(M/M/1) : (N/FCFS)*.

(Single Server / Finite Queue)

Step 2: Given data

Arrival rate λ = 15 minutes.

Service rate μ = 33 minutes.

Step 3: To find the following

- The probability that the yard is empty i.e., P_0
- Average number of trains in the system i.e., L_s

Step 4 : Required computations

- The probability that the yard is empty is given by

$$P_0 = \frac{1-\rho}{1-(\rho)^{N+1}} \quad , \rho = \frac{\lambda}{\mu} = 2.2$$

$$P_0 = \frac{\rho - 1}{\rho^{N+1} - 1}$$

$$= \frac{2.2 - 1}{(2.2)^5 - 1} = \frac{1.2}{50.5}$$

$$= 0.0237$$

Average number of trains in the system $L_s = \sum_{n=0}^{N} nP_n$

$$= P_1 + 2P_2 + 3P_3 + 4P_4$$

$$= \rho P_0 + 2\rho^2 P_0 + 3\rho^3 P_0 + 4\rho^4 P_0$$

$$= P_0 \left[\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 \right]$$

$$= (0.0237) \left[2.2 + 2(2.2)^2 + 3(2.2)^3 + 4(2.2)^4 \right]$$

$$= 3.26$$

Example 3:

A barbershop has space to accommodate only 10 customers . He can serve only one person at a time. If a customer comes to his shop and finds it full he goes to the next shop. Customers randomly arrive at an average rate λ = 10 per hour and the barber service time is exponential with an average of 5 minutes per customer . Find P_0 & P_n

Step 1: Model Identification

Since there is only one barber, the number of service channels is one. Also, since only 10 chairs are available, the capacity of the system is finite. Hence this problem comes under the model

(M/M/1) : (N/FCFS)

(Single Server / Finite Queue)

Step 2: Given data

$$\text{Arrival rate } \lambda = \frac{10}{60} = \frac{1}{6}$$

$$\text{Service rate } \mu = \frac{1}{5}$$

$$\text{Capacity of the system } N = 10$$

Step 3: To find the following

i) P_0

ii) P_n

step 4: Required Computations

Traffic Intensity

$$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$$

$$P_0 = \frac{1-\rho}{1-(\rho)^{N+1}} = \frac{1-\frac{5}{6}}{1-\left(\frac{5}{6}\right)^{11}}$$

$$= \frac{0.1667}{0.8655} = 0.1926$$

$$P_n = \left(\frac{1-\rho}{1-(\rho)^{N+1}} \right) \cdot \rho^N$$

$$= (0.1926)\left(\frac{5}{6}\right)^n \quad (n = 0, 1, 2, 3, \ldots 10)$$

Exercise:

1) A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park is exponentially distributed with mean of 5 hour. How many cars are there in the car park on an average.

[Hint : Model identification – Given N, λ & μ. Find L_s and P_0]

2) In a railway marshalling yard, goods train arrive at the rate of 30 trains per day. Assume that the inter arrival time follows as exponential distribution and the service time is also to be assumed as exponential with mean of 36 minutes. Calculate

- The probability that the yard is empty.
- The average queue length, assuming the line capacity of the yard is 9 trains.

[Hint : Model identification – Given N, λ & μ. Find L_s and P_0]

3) Patients arrival at a clinic having single doctor according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

- Find the effective arrival rate at the clinic.
- What is the probability that an arriving patient will not wait?
- What is the expected waiting time until a patient is discharged from the clinic?

[Hint : Model identification – Given N, λ & μ. Find L_s , P_0, Effective arrival rate]