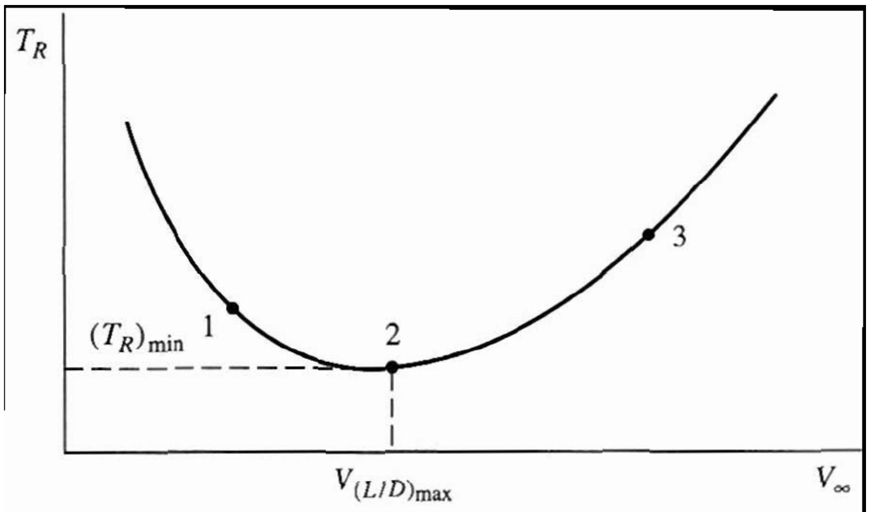
Analytical Approach

For steady, level flight we have

$$T_R = D = \frac{D}{W}W = \frac{D}{L}W$$

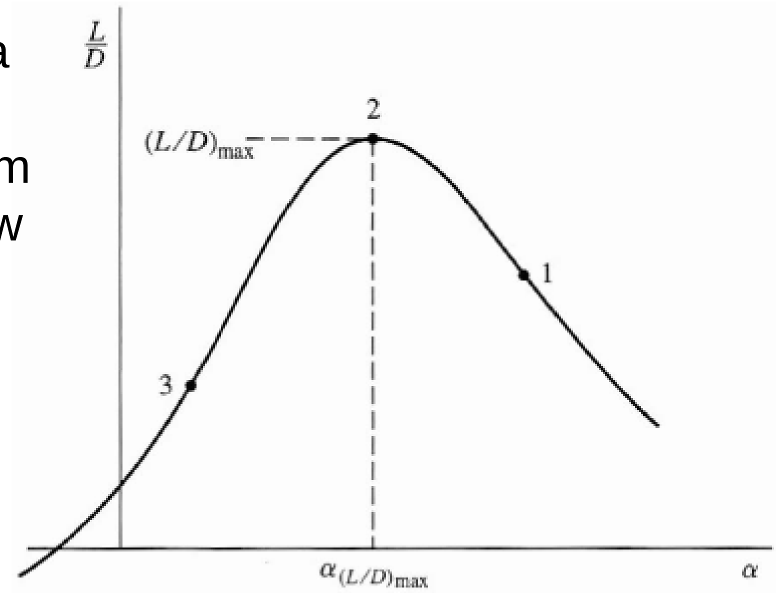


For an airplane with fixed weight, TR decreases as L/D increases. Minimum **TR** occurs when L/D **is** maximum. The lift-to-drag ratio is one of the most important parameters affecting airplane performance. It is a direct measure of the aerodynamic efficiency of an airplane. The lift-to-drag ratio is the same as the ratio of **CL** to CD

$$\frac{L}{D} = \frac{\frac{1}{2}\rho_\infty V_\infty^2 S C_L}{\frac{1}{2}\rho_\infty V_\infty^2 S C_D} = \frac{C_L}{C_D}$$

Since *CL* and ***CD*** are both functions of the angle of attack of the airplane *a,* then *L/D* itself is a function of *alpha.*

Starting at point **3**, *L/D* first increases, reaches a maximum (point 2), and then decreases. TR correspondly first decreases, reaches a minimum and then increases. Point 1 corresponds to a low velocity, with a large angle of attack and with a value of *L/ D* far away from its maximum value. In a TR curve each different point on the curve corresponds to a different angle of attack and a different *L/D.*



Analytical approach

The drag (hence TR) for a given airplane in steady, level flight is a function of altitude (denoted by *h*), velocity, and weight: $D = f(h, V_\infty, W)$

When the altitude h changes, so does density hence *D* changes. Clearly, as *V*, changes, *D* changes. As *W* changes, so does the lift *L;* in turn, the induced drag (drag due to lift) changes, and hence the total drag changes.

$$D = q_\infty S C_D = q_\infty S(C_{D,0} + K C_L^2)$$

$$L = W = q_\infty S C_L = \tfrac{1}{2}\rho_\infty V_\infty^2 S C_L$$

$$C_L = \frac{2W}{\rho_\infty V_\infty^2 S}$$

$$D = \frac{1}{2}\rho_\infty V_\infty^2 S \left[C_{D,0} + 4K \left(\frac{W}{\rho_\infty V_\infty^2 S} \right)^2 \right]$$

Tr=D
In terms of dynamic pressure

$$T_R = q_\infty S C_{D,0} + \frac{K S}{q_\infty} \left(\frac{W}{S} \right)^2$$

rearranging

$$q_\infty^2 S C_{D,0} - q_\infty T_R + K S\left(\frac{W}{S}\right)^2 = 0$$

$$\downarrow$$

$$q_\infty = \frac{T_R \pm \sqrt{T_R^2 - 4SC_{D,0}K(W/S)^2}}{2SC_{D,0}}$$

$$= \frac{T_R/S \pm \sqrt{(T_R/S)^2 - 4C_{D,0}K(W/S)^2}}{2C_{D,0}}$$

$$\downarrow$$

$$V_\infty^2 = \frac{T_R/S \pm \sqrt{(T_R/S)^2 - 4C_{D,0}K(W/S)^2}}{\rho_\infty C_{D,0}}$$

The parameter *TR/S* is analogous to the wing loading *W/S,* the quantity *TR/S* is sometimes called the *thrust loading.* However, T*R/S* is not quite as fundamental as the wing loading *W/S* or the thrust-to-weight ratio *TR/* W. *TR/S* is simply a combination of *TR/ W* and *W/S* via

$$\frac{T_R}{S} = \frac{T_R}{W}\frac{W}{S} \quad \Longrightarrow \quad \boxed{V_\infty = \left[\frac{(T_R/W)(W/S) \pm (W/S)\sqrt{(T_R/W)^2 - 4C_{D,0}K}}{\rho_\infty C_{D,0}}\right]^{1/2}}$$

$$V_\infty = \left[\frac{(T_R/W)(W/S) \pm (W/S)\sqrt{(T_R/W)^2 - 4C_{D,0}K}}{\rho_\infty C_{D,0}} \right]^{1/2}$$

This equation gives the two flight velocities associated with a given value of *TR,* the higher velocity is obtained from the positive discriminant and the lower velocity from the negative discriminant. For a given velocity,*TR* depends on

1. Thrust-to-weight ratio *TR/ W*

2. Wing loading *WIS*

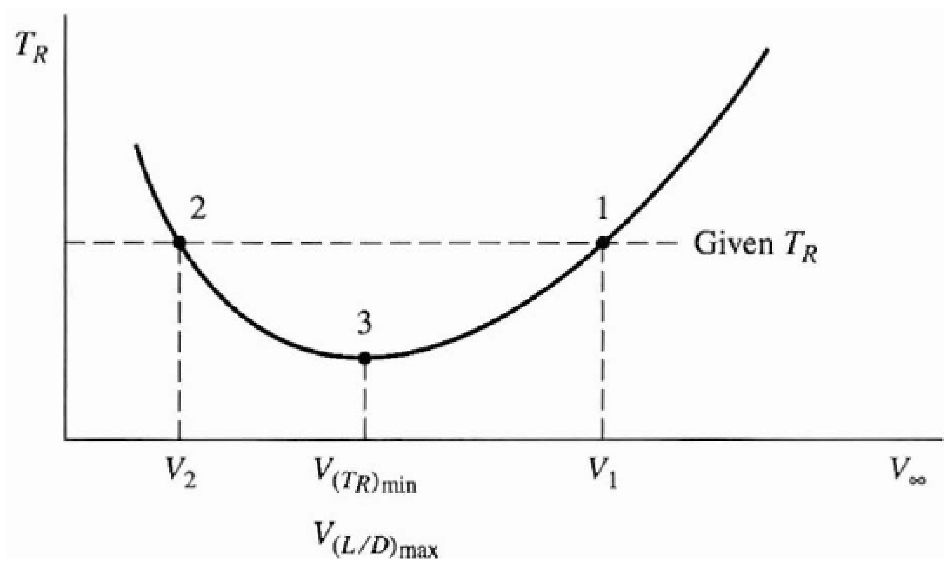
3. The drag polar, that is, *CD0 and* K

V∞ also depends on altitude via the density. When the discriminant equals zero, then only one solution for V, is obtained. This corresponds to the minimum *TR.* That is, when

$$\left(\frac{T_R}{W}\right)^2 - 4C_{D,0}K = 0$$

then

$$V_{(T_R)_{\min}} = \left[\frac{1}{\rho_\infty C_{D,0}} \left(\frac{T_R}{W}\right)_{\min} \frac{W}{S} \right]^{1/2}$$



The value of (TR/W)min is given by

$$\left(\frac{T_R}{W}\right)^2_{\min} = 4C_{D,0}K \longrightarrow \boxed{\left(\frac{T_R}{W}\right)_{\min} = \sqrt{4C_{D,0}K}}$$

The corresponding velocity is

$$V_{(T_R)_{\min}} = \left[\frac{1}{\rho_\infty C_{D,0}}\left(\frac{T_R}{W}\right)_{\min}\frac{W}{S}\right]^{1/2} \longrightarrow V_{(T_R)_{\min}} = \left(\frac{\sqrt{4C_{D,0}K}}{\rho_\infty C_{D,0}}\frac{W}{S}\right)^{1/2}$$

$$\boxed{V_{(T_R)_{\min}} = V_{(L/D)_{\max}} = \left(\frac{2}{\rho_\infty}\sqrt{\frac{K}{C_{D,0}}}\frac{W}{S}\right)^{1/2}}$$

$$\boxed{\left(\frac{T_R}{W}\right)_{\min} = \sqrt{4C_{D,0}K}} \longrightarrow \left(\frac{D}{L}\right)_{\min} = \sqrt{4C_{D,0}K} \longrightarrow \boxed{\left(\frac{L}{D}\right)_{\max} = \frac{1}{\sqrt{4C_{D,0}K}}}$$

the value of $(TR/W)_{min}$ depends ***only*** on the drag polar, that is, the values of $\boldsymbol{Cd_0}$ and K. The velocity for $(TR)_{min}$ depends on the altitude (via the density), the drag polar (via Cd_0 and K), and the wing loading W/S. The airplane weight does not appear separately, but rather always appears as part of a ratio, namely TR/W and W/S. The value of (TR)min is independent of altitude, but that the velocity at which it occurs increases with increasing altitude (decreasing density). The effect of increasing the zero-lift drag coefficient ***Cd0*** is to increase (TR)min and to decrease the velocity at which it occurs. The effect of increasing the drag-due-to-lift factor K (say by decreasing the aspect ratio) is to increase (TR)min and increase the velocity at which it occurs.

The maximum lift-to-drag ratio is ***solely*** dependent on the drag polar. An ***increase*** in the zero-lift drag coefficient ***Cd0 and/***or an increase in the drag-due-to-lift factor K ***decreases*** the value of the maximum lift-to-drag ratio.

$$\boxed{\left(\frac{L}{D}\right)_{max} = \frac{1}{\sqrt{4C_{D,0}K}}}$$

was derived from a consideration of minimizing thrust required in steady, level flight. The same result can be obtained by a simple consideration of the lift-to-drag ratio completely independent of any consideration of ***TR***.
The lift-to-drag ratio is:

$$\frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D,0} + KC_L^2}$$

Differentiating with respect to Cl

$$\frac{d(C_L/C_D)}{dC_L} = \frac{C_{D,0} + KC_L^2 - C_L(2KC_L)}{(C_{D,0} + KC_L^2)^2} = 0$$

when L/D is a maximum value, the zero-lift drag equals the drag due to lift:

$$\boxed{C_{D,0} = KC_L^2}$$

$$C_L = \sqrt{\frac{C_{D,0}}{K}}$$

$$\left(\frac{L}{D}\right)_{max} = \left(\frac{C_L}{C_{D,0} + KC_L^2}\right)_{max}$$

the above derivation made no assumptions about steady, level flight, and no consideration was given to minimum ***TR.*** It is a ***general*** result, having to do with the aerodynamics of the airplane via the drag polar. It is the same result whether the airplane is in climbing flight, turning flight, etc.

$$\boxed{\left(\frac{L}{D}\right)_{max} = \left(\frac{C_L}{C_D}\right)_{max} = \sqrt{\frac{1}{4C_{D,0}K}}}$$

However, the ***velocity*** at which ***(L/ D)max is*** dependent will be different for climbing flight or turning flight compared to steady, level flight. For a steady, level flight L = W

$$L = W = \tfrac{1}{2}\rho_\infty V_\infty^2 S C_L$$

$$\downarrow$$

$$W = \frac{1}{2}\rho_\infty V_{(L/D)_{\max}}^2 S\sqrt{C_{D,0}/K}$$

$$\downarrow$$

$$\boxed{V_{(L/D)_{\max}} = \left(\frac{2}{\rho_\infty}\sqrt{\frac{K}{C_{D,0}}}\frac{W}{S}\right)^{1/2}}$$