

## Particles, Rigid Bodies and “Real Bodies”

Real bodies are normally idealized either as particles or as rigid bodies.

*Particle* - body of negligible dimensions - dimensions of the body are unimportant to the description of its motion – In planetary motion, the planets are considered as particles.

*Rigid body* - body that has a finite size but it does not deform. useful approximation when the deformation of a body is negligible compared to the overall motion – Aircraft and launch vehicle motion

*Real bodies* - finite size and are always deformable under loading – dynamic behavior - mainly used while carrying out detailed design and analysis of structures, navigation, guidance, control systems, vibration analysis and so on.

## NEWTON'S LAWS

1. A particle in isolation moves with constant velocity

A particle in isolation - particle does not interact with any other particle.

Constant velocity - particle moves along a straight line with constant

Speed - can be at rest

Motion (e.g. velocity and acceleration) we observe depends on the reference frame we use - law can not be verified in all reference frames.

The reference frames for which this law is satisfied are called inertial reference frames - Newton's first law postulates that inertial reference frames exist.

2. The acceleration of a particle relative to an inertial reference frame is equal to the force per unit mass applied to the particle.

$F$  represents the (vector) sum of all forces acting on a particle of mass ' $m$ ', any inertial observer will see that the particle has an acceleration ' $a$ ' which is given by,

$$F = m \cdot a \quad (1)$$

Equation (1) is a vector equation - force and the acceleration always have the same direction and the ratio of their magnitudes is ' $m$ '.

3. The forces of action and reaction between interacting bodies are equal in magnitude and opposite in direction

Clearly satisfied when the bodies are in contact and in static equilibrium  
Situation for bodies in motion interacting at a distance - e.g.

electromagnetic or gravity interactions - Newton's third law breaks down  
Electromagnetic signals travel at a finite speed and therefore there is a time delay whenever two bodies interact at a distance

Error made by assuming that these interactions are instantaneous is  
Negligible - Newton's third law is applicable in many cases

# LAW OF UNIVERSAL ATTRACTION

Force of attraction between any two particles, of masses  $M$  and  $m$ , respectively, has a magnitude,  $F$ , given by

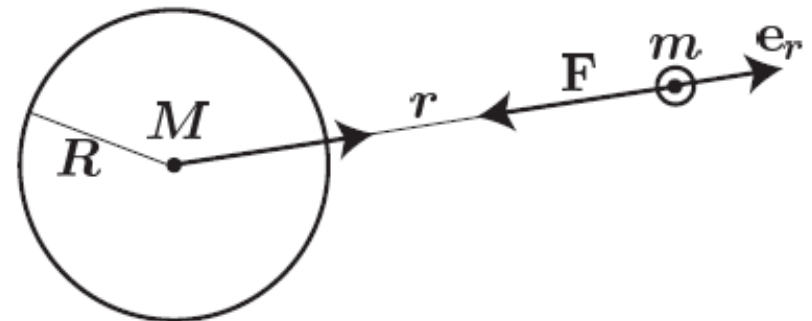
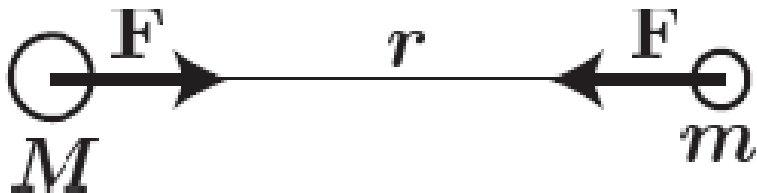
$$F = GMm / r^2 \quad (2)$$

where  $r$  is the distance between the two particles, and

$G = 6.673 \times 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2)$  is the universal constant of gravitation

law of gravitation is strictly valid for point masses

When the size of the masses is comparable to the distance between the masses one would observe deviations to the above law. It turns out that if the mass  $M$  is distributed uniformly over a sphere of radius  $R$ , the force on a mass  $m$ , outside  $M$ , is still given by (2), with  $r$  being measured from the sphere's center.



## WEIGHT

The gravitational attraction from the earth to any particle located near the surface of the earth

weight,  $W$  of mass  $m$  at sea level

$$\mathbf{W} = -G \frac{M_e m}{R_e^2} \mathbf{e}_r = -g_0 m \mathbf{e}_r = m \mathbf{g}_0$$

$M_e \approx 5.976 \times 10^{24}$  kg and  $R_e \approx 6.371 \times 10^6$  m, are the mass and radius of the earth, respectively, and  $\mathbf{g}_0 = -(GM_e/R_e^2) \mathbf{e}_r$  is the gravitational acceleration vector at sea level. The average value of its magnitude is  $g_0 = 9.825$  m/s<sup>2</sup>.

weight at an altitude  $h$  above sea level is given by

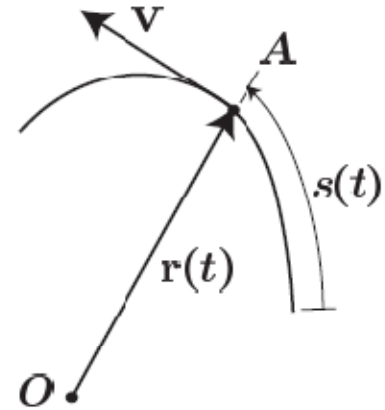
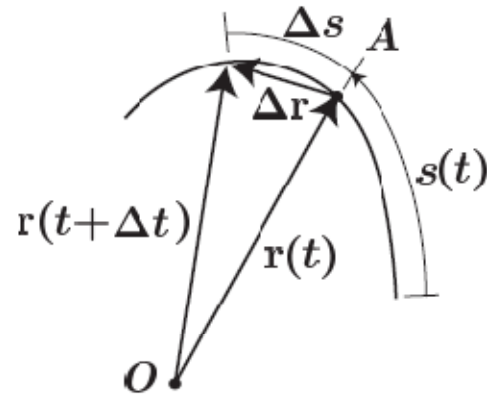
$$\mathbf{W} = -G \frac{M_e m}{(R_e + h)^2} \mathbf{e}_r = -g_0 \frac{R_e^2}{(R_e + h)^2} m \mathbf{e}_r = m \frac{R_e^2}{(R_e + h)^2} \mathbf{g}_0 .$$

Earth is not quite spherical - weight does not exactly obey the inverse – squared law -  $g_0$ , at the poles and at the equator, is slightly different.

Earth is also rotating - introduces an inertial centrifugal force which has the effect of reducing the vertical component of the weight.

# CURVILINEAR MOTION - CARTESIAN COORDINATES

## Velocity Vector



Average velocity of the particle over this small increment of time

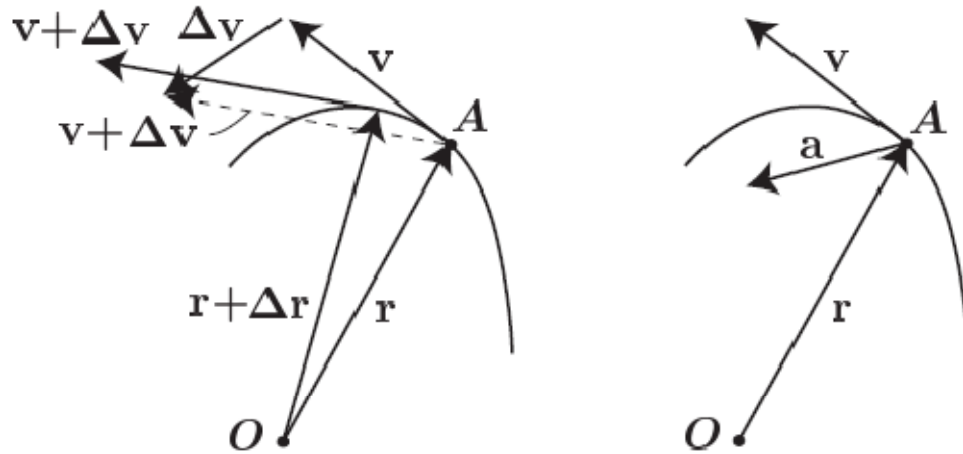
$$\mathbf{v}_{ave} = \frac{\Delta \mathbf{r}}{\Delta t}$$

Instantaneous velocity vector

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} \equiv \frac{d\mathbf{r}(t)}{dt} \equiv \dot{\mathbf{r}}, \quad \text{always tangent to the path}$$

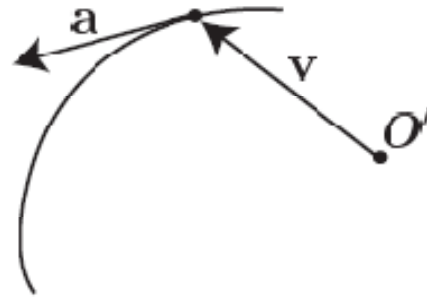
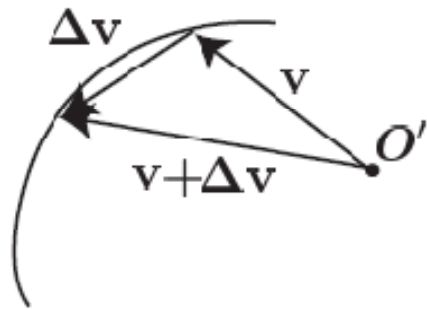
# Acceleration Vector

$$a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} \equiv \frac{dv}{dt} = \frac{d^2 r}{dt^2} .$$



The acceleration vector will, in general, not be tangent to the trajectory (in fact it is only tangent when the velocity vector does not change direction).

To visualize the acceleration vector is to translate the velocity vectors, at different times, such that they all have a common origin, say,  $O'$ . Then, the heads of the velocity vector will change in time and describe a curve in space called the *hodograph*



Acceleration vector is tangent to the hodograph at every point

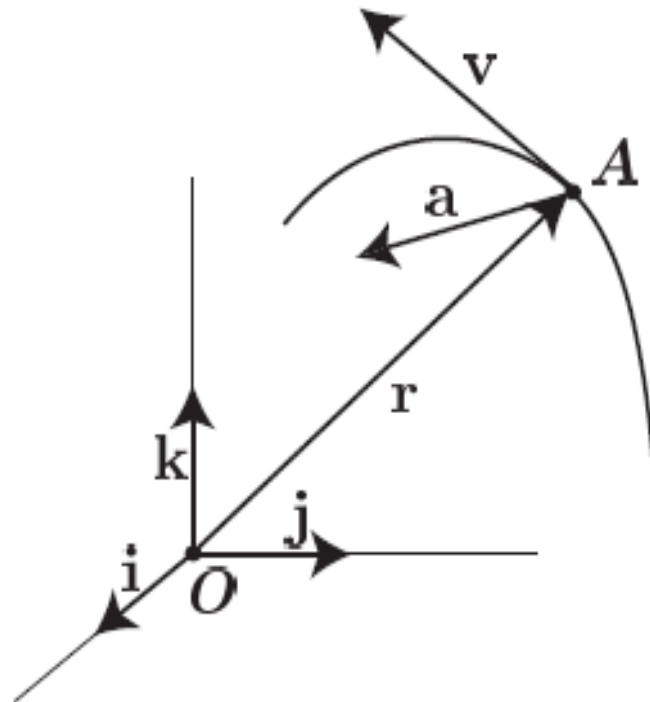


## Cartesian Coordinates

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k} = \dot{\mathbf{r}}(t)$$

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k} = \dot{v}_x(t)\mathbf{i} + \dot{v}_y(t)\mathbf{j} + \dot{v}_z(t)\mathbf{k} = \dot{\mathbf{v}}(t)$$



speed is given by  $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ , and the magnitude of the acceleration is  $a = \sqrt{a_x^2 + a_y^2 + a_z^2}$ .

# Equations of Motion in Cartesian Coordinates

Rectangular Cartesian coordinates,  $xyz$ ,  $F = F_x i + F_y j + F_z k$  and  $a = a_x i + a_y j + a_z k$ .

$$F_x = ma_x = m\ddot{x}$$

In component form

$$F_y = ma_y = m\ddot{y}$$

$$F_z = ma_z = m\ddot{z}$$

The initial position  $r_0$  and velocity  $v_0$ .

$$r(0) = x(0)i + y(0)j + z(0)k = r_0,$$

$$v(0) = \dot{x}(0)i + \dot{y}(0)j + \dot{z}(0)k = v_0.$$

## Analytical Integration – Some General Cases

Analytical solution is possible - F is either constant or depends on t only

$$v_x(t) = v_x(0) + \frac{1}{m} \int_0^t F_x(t) dt, \quad v_y(t) = v_y(0) + \frac{1}{m} \int_0^t F_y(t) dt, \quad v_z(t) = v_z(0) + \frac{1}{m} \int_0^t F_z(t) dt,$$

and the position,

$$x(t) = x(0) + \frac{1}{m} \int_0^t v_x(t) dt, \quad y(t) = y(0) + \frac{1}{m} \int_0^t v_y(t) dt, \quad z(t) = z(0) + \frac{1}{m} \int_0^t v_z(t) dt.$$

## Numerical Integration

The most general way of solving the equations of motion is by numerical integration. In this case we do not compute the solution of the problem but an approximation to it. It is typically useful to work only with first order equations, so we write,

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

$$\dot{z} = v_z$$

This is a set of 6 first order ODE's,  
with initial conditions

$$x(0) = x_0$$

$$y(0) = y_0$$

$$z(0) = z_0$$

$$v_x(0) = v_{x0},$$

$$v_y(0) = v_{y0},$$

$$v_z(0) = v_{z0} .$$

$$\dot{v}_x = \frac{1}{m} F_x(x, y, z, v_x, v_y, v_z, t)$$

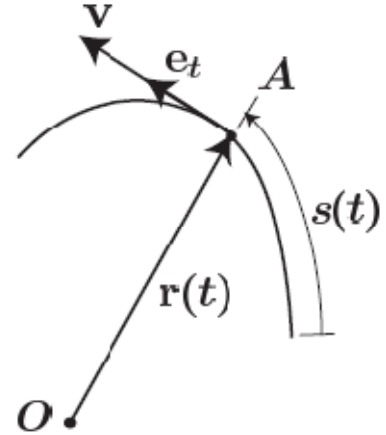
$$\dot{v}_y = \frac{1}{m} F_y(x, y, z, v_x, v_y, v_z, t)$$

$$\dot{v}_z = \frac{1}{m} F_z(x, y, z, v_x, v_y, v_z, t)$$

# Intrinsic Coordinates - Tangential, Normal and Bi-normal components

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = v\mathbf{e}_t,$$

where  $\mathbf{r}(t)$  is the position vector,  $v = s'$  is the speed,  $\mathbf{e}_t$  is the unit tangent vector to the trajectory, and  $s$  is the path coordinate along the trajectory.



The unit tangent vector can be written as,

$$\mathbf{e}_t = \frac{d\mathbf{r}}{ds}.$$

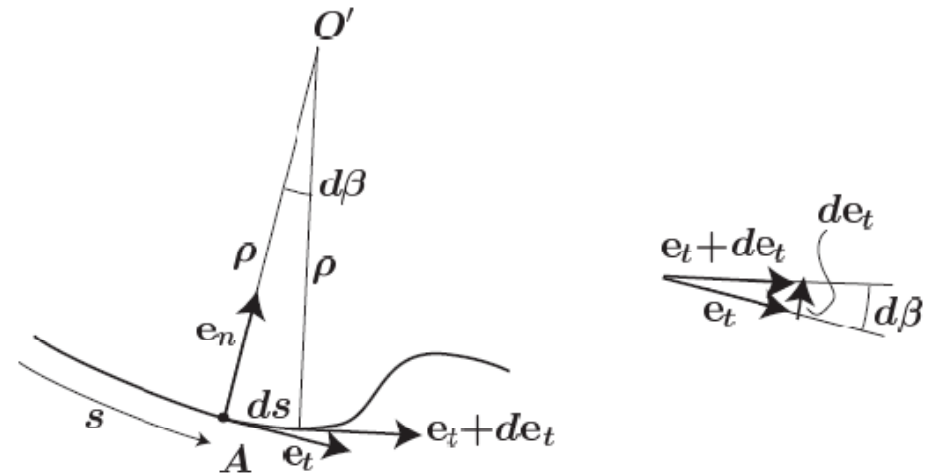
Acceleration vector is the derivative of the velocity vector with respect to time

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt}.$$

vector  $\mathbf{e}_t$  is the local unit tangent vector to the curve which changes from point to point. Consequently, the time derivative of  $\mathbf{e}_t$  will, in general, be nonzero.

$$\frac{d\mathbf{e}_t}{dt} = \frac{d\mathbf{e}_t}{ds} \frac{ds}{dt} = \frac{d\mathbf{e}_t}{ds} v$$

In order to calculate the derivative of  $e_t$ , we note that, since the magnitude of  $e_t$  is constant and equal to one, the only changes that  $e_t$  can have are due to rotation, or swinging.



When we move from  $s$  to  $s + ds$ , the tangent vector changes from  $e_t$  to  $e_t + de_t$ . The change in direction can be related to the angle  $d\beta$ .

The direction of  $de_t$ , which is perpendicular to  $e_t$ , is called the normal direction. On the other hand, the magnitude of  $de_t$  will be equal to the length of  $e_t$  (which is one), times  $d\beta$ . Thus, if  $e_n$  is a unit normal vector in the direction of  $de_t$ , we can write

$$de_t = d\beta e_n$$

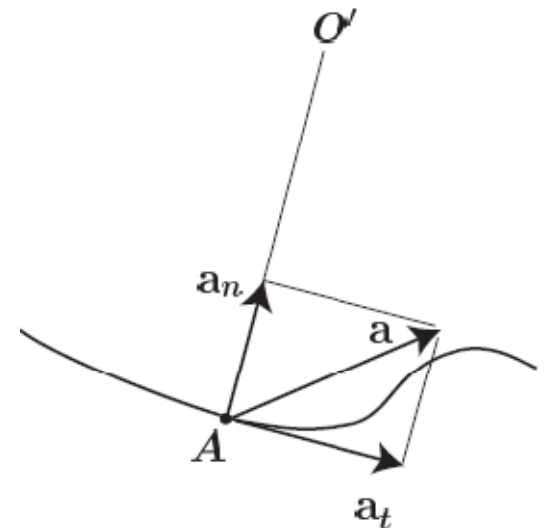
Dividing by  $ds$  yields, 
$$\frac{de_t}{ds} = \frac{d\beta}{ds} e_n = \kappa e_n = \frac{1}{\rho} e_n$$

Here,  $\kappa = d\beta/ds$  is a local property of the curve, called the curvature, and  $\rho = 1/\kappa$  is called the radius of curvature.

we have that 
$$\frac{de_t}{dt} = \frac{d\beta}{ds} v e_n = \dot{\beta} e_n = \frac{v}{\rho} e_n .$$

The acceleration can be written as 
$$a = \frac{dv}{dt} e_t + \frac{v^2}{\rho} e_n = a_t e_t + a_n e_n .$$

$a_t = v\dot{\phantom{v}}$ , is the tangential component of the acceleration, and  $a_n = v^2/\rho$ , is the normal component of the acceleration.  $a_n$  is the component of the acceleration pointing towards the center of curvature, it is sometimes referred to as centripetal acceleration.



When  $a_t$  is nonzero, the velocity vector changes magnitude, or stretches.  
When  $a_n$  is nonzero, the velocity vector changes direction, or swings.  
The modulus of the total acceleration can be calculated as

$$a = \sqrt{a_t^2 + a_n^2}.$$

The vectors  $e_t$  and  $e_n$ , and their respective coordinates  $t$  and  $n$ , define two orthogonal directions. The plane defined by these two directions, is called the *osculating plane*. This plane changes from point to point, and can be thought of as the plane that locally contains the trajectory (Tangent is the current direction of the velocity, and the normal is the direction into which the velocity is changing).

Define a right handed set of axes - an additional unit vector which is orthogonal to  $e_t$  and  $e_n$ . This vector is called the bi-normal, and is defined as  $e_b = e_t \times e_n$ .

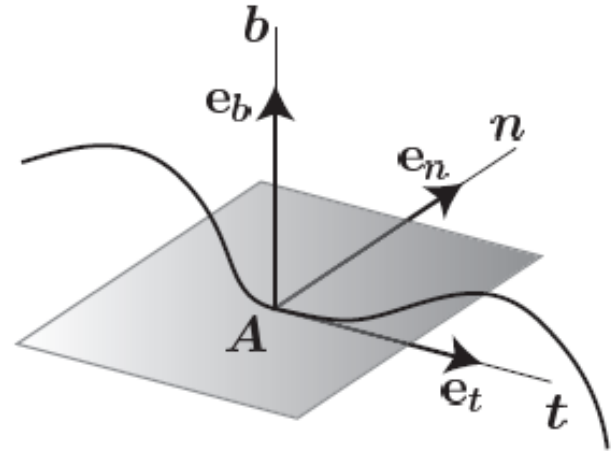


At any point in the trajectory, the position vector, the velocity and acceleration can be referred to these axes.

velocity and acceleration take very simple forms

$$\mathbf{v} = v\mathbf{e}_t$$

$$\mathbf{a} = \dot{v}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n .$$



Component of the acceleration along the binormal is always zero. When the trajectory is planar, the binormal stays constant (orthogonal to the plane of motion). However, when the trajectory is a space curve, the binormal changes with  $s$ . Derivative of the binormal is always along the direction of the normal. The rate of change of the binormal with  $s$  is called the torsion,  $\tau$ .

$$\frac{de_b}{ds} = -\tau e_n \quad \text{or,} \quad \frac{de_b}{dt} = -\tau v e_n.$$

Whenever the torsion is zero, the trajectory is planar, and whenever the curvature is zero, the trajectory is linear

## Equations of Motion

In tangent, normal and bi-normal components, t-n-b, we write

$$F = F_t e_t + F_n e_n \quad \text{and} \quad a = a_t e_t + a_n e_n$$

positive direction of the normal coordinate is that pointing towards the center of curvature

$$F_t = m a_t = m \dot{v} = m \ddot{s}$$

$$F_n = m a_n = m \frac{v^2}{\rho}$$

component of the acceleration along the bi-normal direction,  $e_b$ , is always zero. Consequently the bi-normal component of the force must also be zero

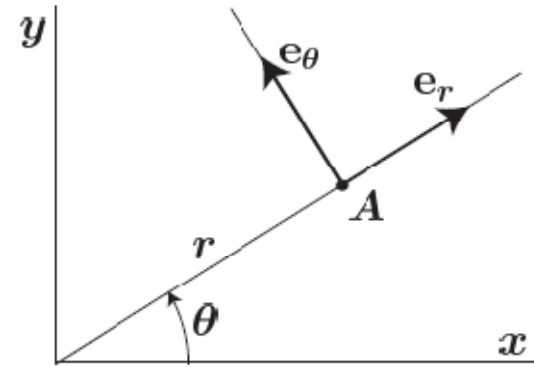
## Other Coordinates Systems

### Polar Coordinates ( $r - \theta$ )

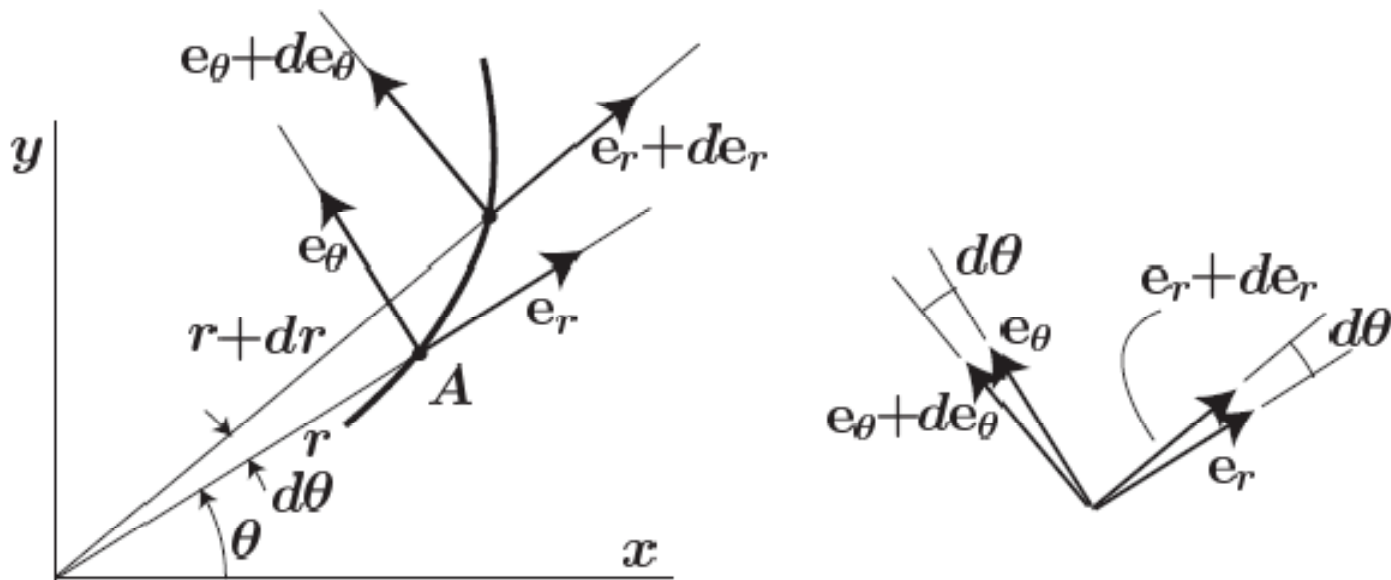
trajectory of a particle will be determined if we know  $r$  and  $\theta$  as a function of  $t$ , i.e.  $r(t), \theta(t)$ . The directions of increasing  $r$  and  $\theta$  are defined by the orthogonal unit vectors  $e_r$  and  $e_\theta$ .

position vector of a particle has a magnitude equal to the radial distance, and a direction determined by  $e_r$ .

$$\mathbf{r} = r \mathbf{e}_r$$



Since the vectors  $e_r$  and  $e_\theta$  are clearly different from point to point, their variation will have to be considered when calculating the velocity and acceleration. Over an infinitesimal interval of time  $dt$ , the coordinates of point A will change from  $(r, \theta)$ , to  $(r + dr, \theta + d\theta)$



The vectors  $e_r$  and  $e_\theta$  do not change when the coordinate  $r$  changes. Thus,  $de_r/dr = 0$  and  $de_\theta/dr = 0$ . On the other hand, when  $\theta$  changes to  $\theta + d\theta$ , the vectors  $e_r$  and  $e_\theta$  are rotated by an angle  $d\theta$ . From the diagram, we see that  $de_r = d\theta e_\theta$ , and that  $de_\theta = -d\theta e_r$ . This is because their magnitudes in the limit are equal to the unit vector as radius times  $d\theta$  in radians. Dividing through by  $d\theta$ , we have,

$$\frac{de_r}{d\theta} = e_\theta, \quad \text{and} \quad \frac{de_\theta}{d\theta} = -e_r$$

Multiplying these expressions by  $d\theta/dt \equiv \dot{\theta}$ ,

we obtain, 
$$\frac{de_r}{d\theta} \frac{d\theta}{dt} \equiv \frac{de_r}{dt} = \dot{\theta} e_\theta, \quad \text{and} \quad \frac{de_\theta}{dt} = -\dot{\theta} e_r .$$

Alternative calculation of the unit vector derivatives

$$e_r = \cos \theta i + \sin \theta j$$

$$e_\theta = -\sin \theta i + \cos \theta j .$$

$$\frac{de_r}{dr} = 0,$$

$$\frac{de_\theta}{dr} = 0,$$

$$\frac{de_r}{d\theta} = -\sin \theta i + \cos \theta j \equiv e_\theta$$

$$\frac{de_\theta}{d\theta} = -\cos \theta i - \sin \theta j \equiv -e_r$$

## Velocity vector

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_r ,$$

$$\mathbf{v} = \dot{r} \mathbf{e}_r + r\dot{\theta} \mathbf{e}_\theta .$$

$v_r = \dot{r}$  is the radial velocity component, and  $v_\theta = r \dot{\theta}$  is the circumferential velocity component. We also have that

$$v = \text{SQRT} (v_r^2 + v_\theta^2)$$

The radial component is the rate at which  $r$  changes magnitude, or stretches, and the circumferential component, is the rate at which  $r$  changes direction, or swings.

## Acceleration vector

Differentiating again with respect to time, we obtain the acceleration

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r} \mathbf{e}_r + \dot{r} \dot{\mathbf{e}}_r + \dot{r} \dot{\theta} \mathbf{e}_\theta + r \ddot{\theta} \mathbf{e}_\theta + r \dot{\theta} \dot{\mathbf{e}}_\theta$$

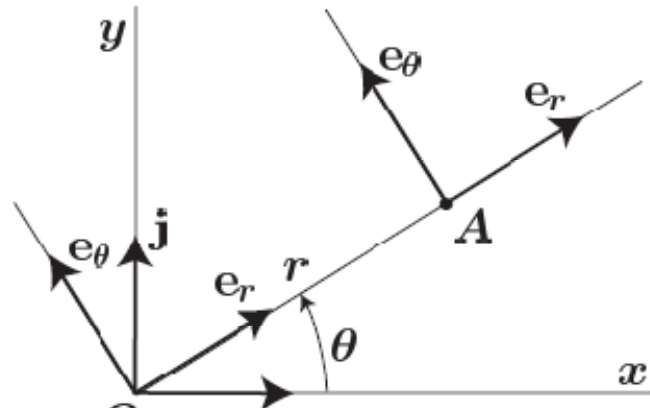
$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2) \mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \mathbf{e}_\theta$$

$a_r = (\ddot{r} - r\dot{\theta}^2)$  is the radial acceleration component

$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$  is the circumferential acceleration component

# Change of basis

Polar coordinates to Cartesian coordinates and vice versa.



$$\begin{pmatrix} A_r \\ A_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix}$$

## Equations of Motion

$$F_r = m a_r = m (\ddot{r} - r\dot{\theta}^2)$$

$$F_\theta = m a_\theta = m (r\ddot{\theta} + 2\dot{r}\dot{\theta}) .$$



# Relative Motion

Types of observers - three different types (or reference frames) depending on their motion with respect to a fixed frame:

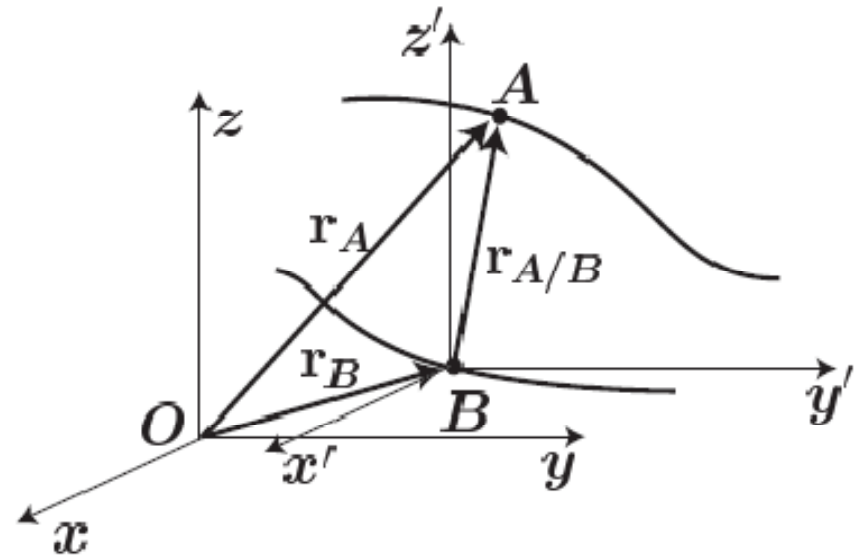
- observers who do not accelerate or rotate, i.e. those who at most have constant velocity.
- observers who accelerate but do not rotate
- observers who accelerate and rotate

## Relative motion using translating axes

consider a fixed reference frame  $xyz$  with origin  $O$  and with unit vectors  $i, j$  and  $k$ . Consider another translating reference frame attached to particle  $B$ ,  $x'y'z'$ , with unit vectors  $i', j'$  and  $k'$ . Angles between the axes  $xyz$  and  $x'y'z'$  do not change during the motion.

position vector  $\mathbf{r}_{A/B}$  defines the position of A with respect to point B in the reference frame  $x'y'z'$ .

The subscript notation “A/B” means “A relative to B”. The positions of A and B relative to the absolute frame are given by the vectors  $\mathbf{r}_A$  and  $\mathbf{r}_B$ , respectively. Thus, we have



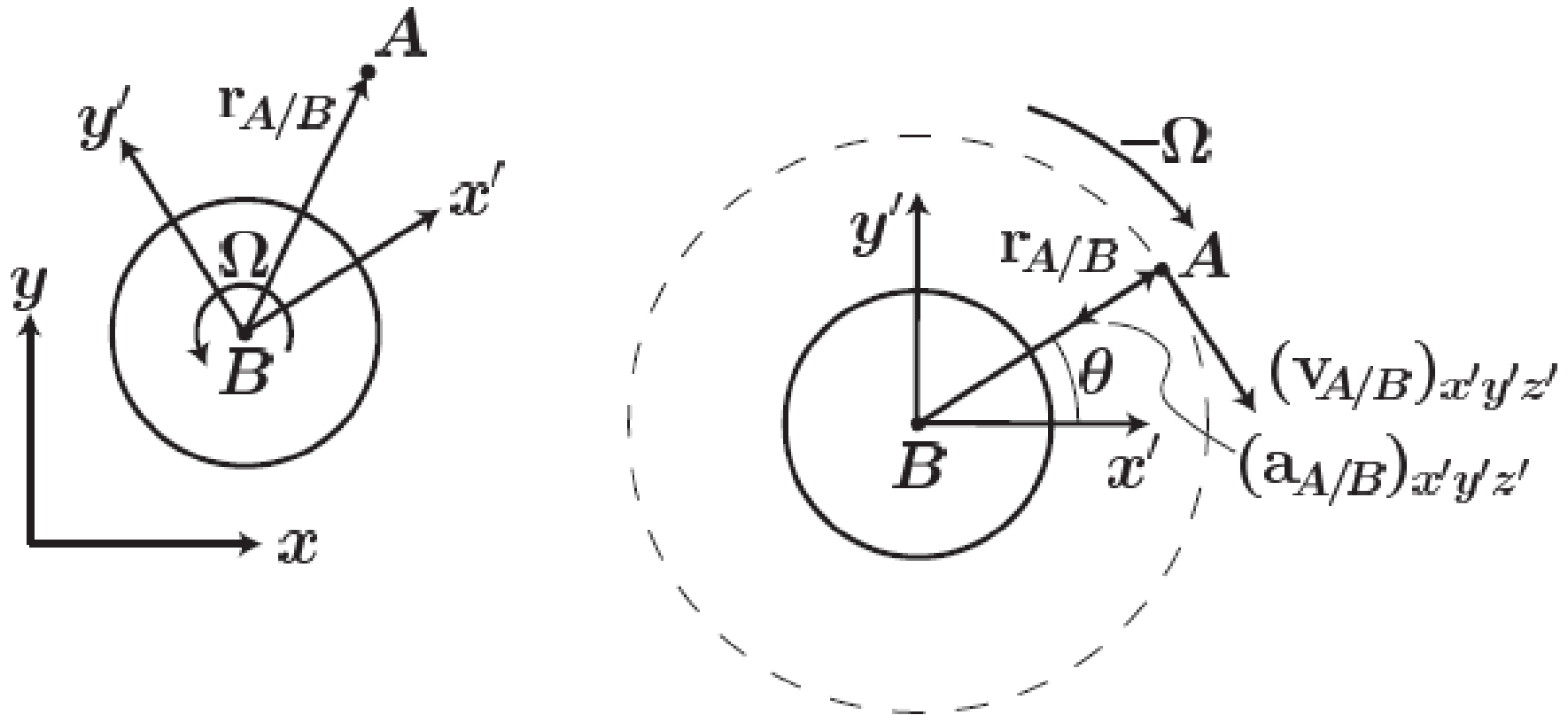
$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B} .$$

$$\ddot{\mathbf{r}}_A = \ddot{\mathbf{r}}_B + \ddot{\mathbf{r}}_{A/B} \quad \text{OR} \quad \mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_{A/B} \quad \text{OR} \quad \mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

## Relative Motion using Translating/Rotating Axes

A particle at rest with respect to the fixed frame  $xy$ , i.e.  $v_A = 0$ ,  $a_A = 0$ , is observed by an observer,  $B$ , who is standing at the center of a turn table. The table rotates with a constant angular velocity of  $\Omega$  rad/s.



Assuming that the platform only rotates about its center and does not move translationally, the position of B will not change, and therefore  $v_B = 0$  and  $a_B = 0$ .

Relative to B, A will not be at rest.

B will see A rotating about B with a constant angular velocity of  $-\Omega$ .

Motion of A as observed by B (rotating with  $x'y'$ )  
velocity and acceleration in local polar coordinates

$$(v_{A/B})_{x'y'} = (\dot{r}_{A/B})_{x'y'} = -\Omega r_{A/B} e_\theta$$

$$(a_{A/B})_{x'y'} = (\dot{v}_{A/B})_{x'y'} = -\frac{(v_{A/B})_{x'y'}}{r_{A/B}} e_r = -\Omega^2 r_{A/B} e_r .$$

$(\cdot)_{x'y'}$ , is used to indicate the velocity, or acceleration, experienced by an observer that rotates with the axes  $x'y'$ .

Assumption - The directions  $x'y'$ , as seen by a rotating observer do not change.

$v_{A/B}$  and  $a_{A/B}$  denote the relative velocity and acceleration of A with respect to B, experienced by a non-rotating observer.

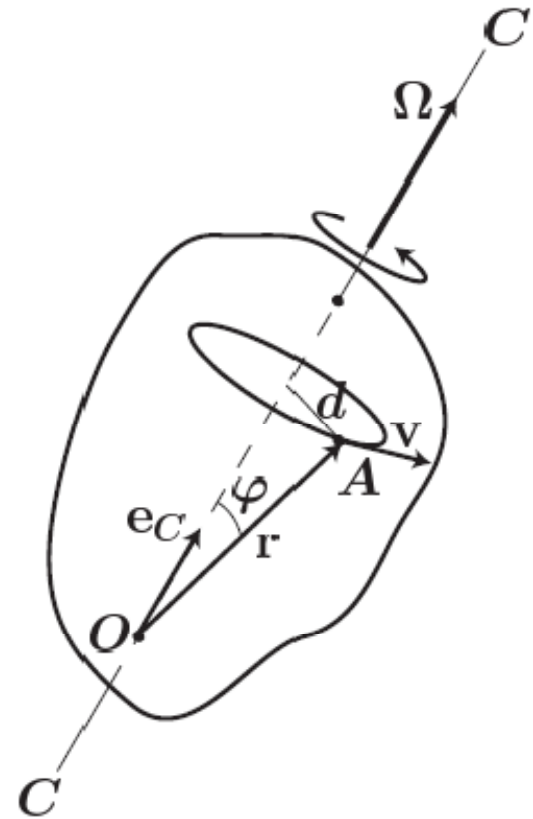
$$v_{A/B} = \dot{r}_{A/B} = 0 \quad a_{A/B} = \dot{v}_{A/B} = 0.$$

## Angular velocity and angular acceleration vectors

Let us consider a rigid body, which is spinning about an axis C – C with an angular velocity of ' $\Omega$  rad/s.

consider a unit vector,  $e_C$ , along the direction of the axis, and define the angular velocity vector, ' $\Omega$ ', as a vector having magnitude ' $\Omega$ ' and direction  $e_C$ .

$$\Omega = \Omega e_C$$

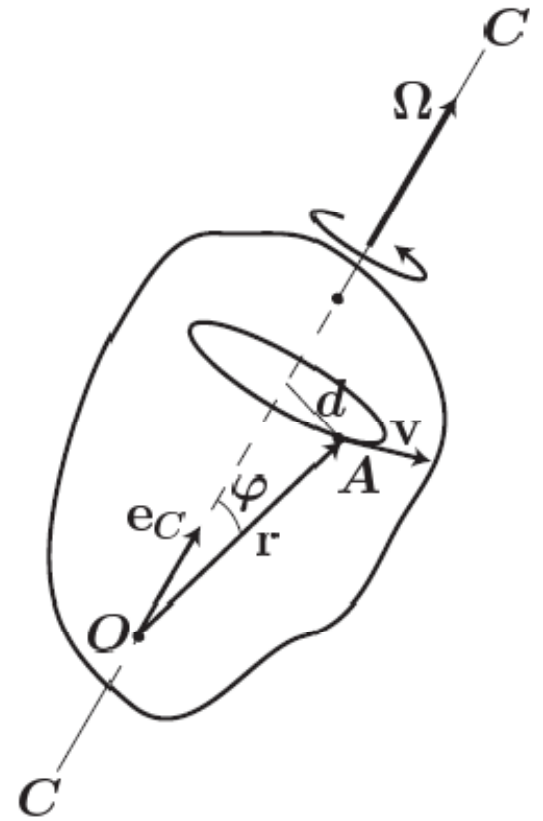


the convention between the direction of rotation and that of  $\boldsymbol{\Omega}$  is determined by the right hand rule. If the body were to rotate in the direction opposite to that shown in the diagram, then we would simply have  $\boldsymbol{\Omega} = -\boldsymbol{\Omega} e_C$ .

Angular velocity vector is useful to express the velocity due to rotation of any point A in the rigid body. Let  $\mathbf{r}$  be the position vector of A relative to an origin point, O, located on the axis of rotation. It turns out that the velocity of A,  $\mathbf{v}$ , can be simply expressed as

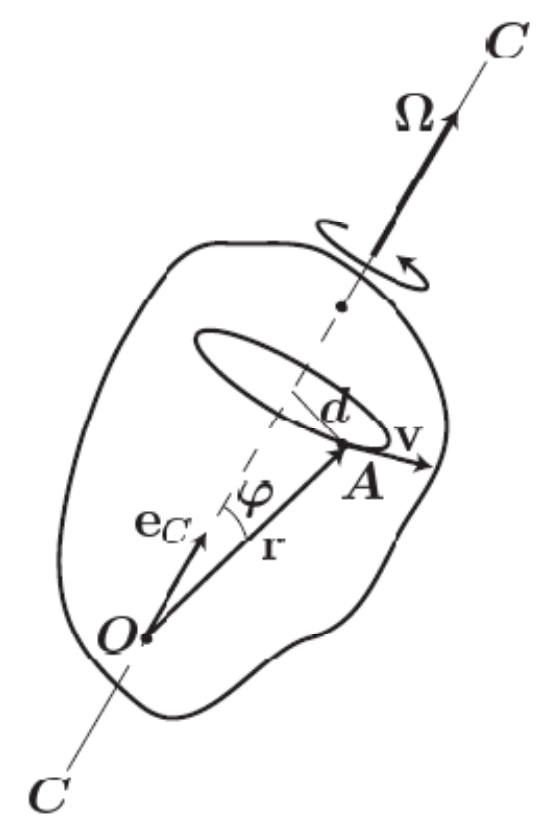
$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$$

HOW?



$$\mathbf{v} = \dot{\boldsymbol{\Omega}} \times \mathbf{r}$$

When the body spins, A describes a circular trajectory around the axis of radius  $d = r \sin \varphi$ , where  $\varphi$  is the angle between  $\mathbf{r}$  and the axis of rotation. Since the body is spinning at a rate of  $\dot{\boldsymbol{\Omega}}$  rad/s, the magnitude of the velocity vector will be  $v = \dot{\boldsymbol{\Omega}} d = r \sin \varphi$ . The direction of the velocity vector  $\mathbf{v}$  will be tangent to the trajectory at A, which means that it will be perpendicular to  $\mathbf{r}$  and  $\mathbf{e}_C$ . By the definition of the vector product, the above is satisfied.



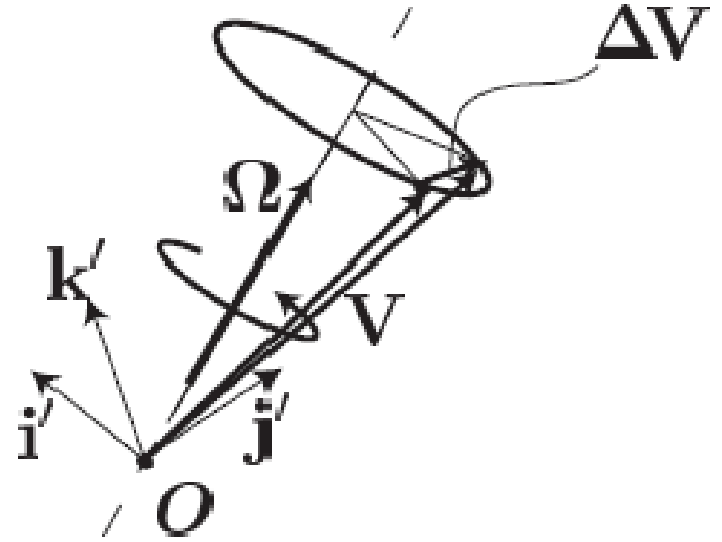
angular acceleration vector

$$\dot{\boldsymbol{\Omega}} = \frac{d\boldsymbol{\Omega}}{dt}$$

# Time derivative of a fixed vector in a rotating frame

We consider a reference frame  $x'y'z'$  rotating with an angular velocity with respect to a fixed frame  $xyz$ .

Let  $V$  be any vector, which is constant relative to the frame  $x'y'z'$ . That is, the vector components in the  $x'y'z'$  frame do not change, and, as a consequence,  $V$  rotates as if it were rigidly attached to the frame.



In the absolute frame, the time derivative will be equal to

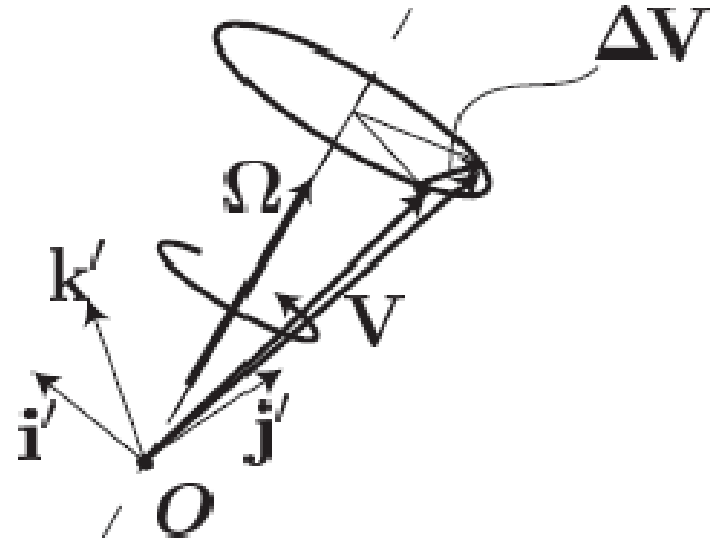
$$\frac{dV}{dt} = \lim_{\Delta t \rightarrow 0} \frac{(V + \Delta V) - V}{\Delta t} = \Omega \times V$$



The above expression applies to any vector which is rigidly attached to the frame  $x'y'z'$ . In particular, it applies to the unit vectors  $i'$ ,  $j'$  and  $k'$ .

Therefore, we have that

$$\begin{aligned}\frac{di'}{dt} &= \boldsymbol{\Omega} \times i' \\ \frac{dj'}{dt} &= \boldsymbol{\Omega} \times j' \\ \frac{dk'}{dt} &= \boldsymbol{\Omega} \times k' .\end{aligned}$$



## Time derivative of a vector in a rotating frame: Coriolis' theorem

Let  $\mathbf{V}$  be an arbitrary vector (e.g. velocity, magnetic field, force, etc.), which is allowed to change in both the fixed  $xyz$  frame and the rotating  $x'y'z'$  frame.

vector  $\mathbf{V}$  in the  $x'y'z'$  frame  $\mathbf{V} = V_{x'}\mathbf{i}' + V_{y'}\mathbf{j}' + V_{z'}\mathbf{k}'.$

The time derivative of  $\mathbf{V}$ , as seen by the fixed frame

$$\begin{aligned}\dot{\mathbf{V}} = \frac{d\mathbf{V}}{dt} &= \dot{V}_{x'}\mathbf{i}' + \dot{V}_{y'}\mathbf{j}' + \dot{V}_{z'}\mathbf{k}' + V_{x'}\dot{\mathbf{i}}' + V_{y'}\dot{\mathbf{j}}' + V_{z'}\dot{\mathbf{k}}' \\ &= (\dot{\mathbf{V}})_{x'y'z'} + \boldsymbol{\Omega} \times \mathbf{V} .\end{aligned}$$

$\dot{V}_{x'}i' + \dot{V}_{y'}j' + \dot{V}_{z'}k' = (\dot{V})_{x'y'z'}$  is the time derivative of the vector  $V$  as seen by the rotating frame.

Hence, for this derivative, the vectors  $i'$ ,  $j'$  and  $k'$  remain unchanged.

$$V_{x'}\dot{i}' + V_{y'}\dot{j}' + V_{z'}\dot{k}' = V_{x'}(\boldsymbol{\Omega} \times i') + V_{y'}(\boldsymbol{\Omega} \times j') + V_{z'}(\boldsymbol{\Omega} \times k') = \boldsymbol{\Omega} \times V$$

This is the change in  $V$  due to the rotation

$$\left(\dot{\quad}\right)_{xyz} = \left(\dot{\quad}\right)_{x'y'z'} + \boldsymbol{\Omega} \times (\quad)$$

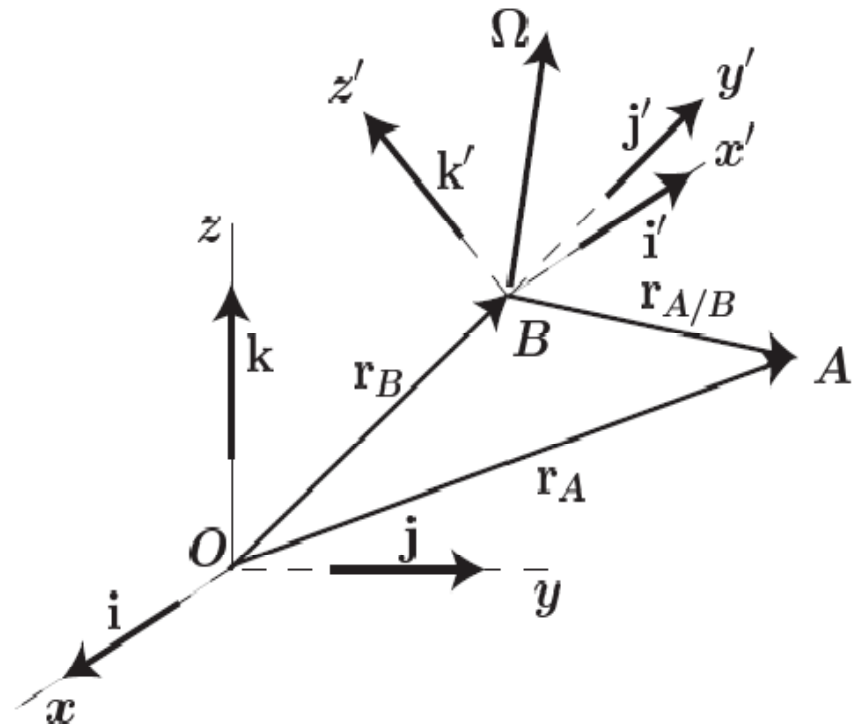
The above expression is known as Coriolis' theorem. Given an arbitrary vector, it relates the derivative of that vector as seen by a fixed frame with the derivative of the same vector as seen by a rotating frame

## Relative Motion using translating / rotating axes

we consider the relationship between the motion seen by an observer B that may be accelerating as well as rotating, and the motion seen by a fixed observer O. Let  $a_B$  be the acceleration of B with respect to O, and let  $\Omega$  and  $\dot{\Omega}$  denote the angular velocity and angular acceleration, respectively, of the frame  $x'y'z'$  rigidly attached to B.

Vectors  $i, j$  and  $k$  are the unit vectors corresponding to the fixed frame  $xyz$ ,

$i', j'$  and  $k'$  - unit vectors corresponding to the rotating frame  $x'y'z'$ .



The position vector of A with respect to the fixed frame at any instance is

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

$\mathbf{r}_A$  and  $\mathbf{r}_B$  expressed in xyz components, and  $\mathbf{r}_{A/B}$  in  $x'y'z'$  components

Velocity vector

$$\begin{aligned} \mathbf{v}_a &= \dot{\mathbf{r}}_B + (\dot{\mathbf{r}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} \\ &= \mathbf{v}_B + (\mathbf{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \mathbf{r}_{A/B} . \end{aligned}$$

Using Coriolis' theorem the derivative of  $\mathbf{r}_{A/B}$

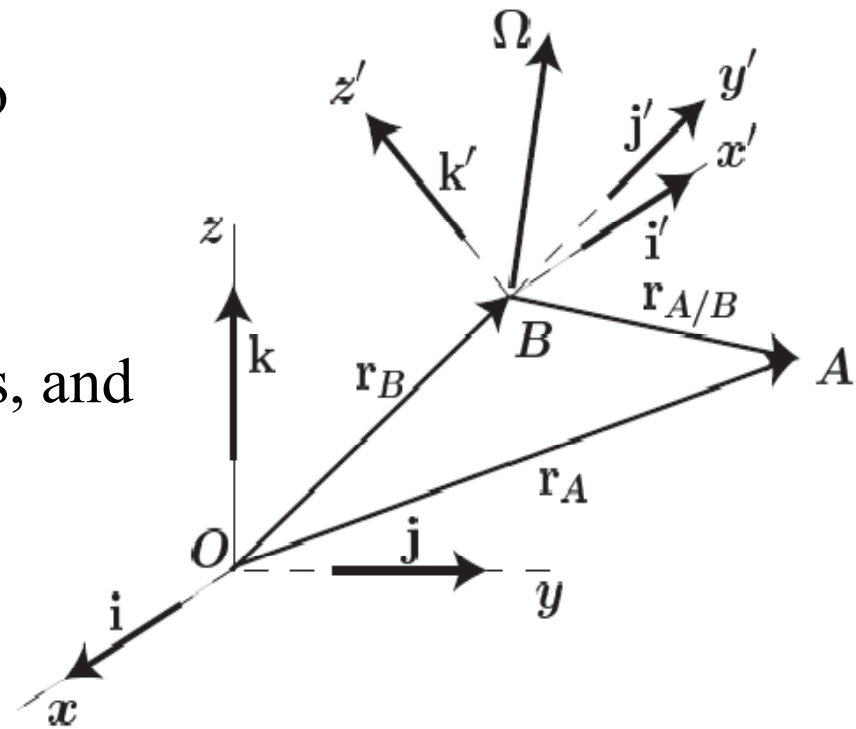
$$\dot{\mathbf{r}}_{A/B} = (\dot{\mathbf{r}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \mathbf{r}_{A/B}$$

$\mathbf{v}_A$  and  $\mathbf{v}_B$  - velocities of A and B, relative to the fixed frame.

$(\mathbf{v}_{A/B})_{x'y'z'}$  is the velocity of A measured by the rotating observer, B.

$\boldsymbol{\Omega}$  - angular velocity of the rotating frame,

$\mathbf{r}_{A/B}$  is the relative position vector of A with respect to B.



## Acceleration vector

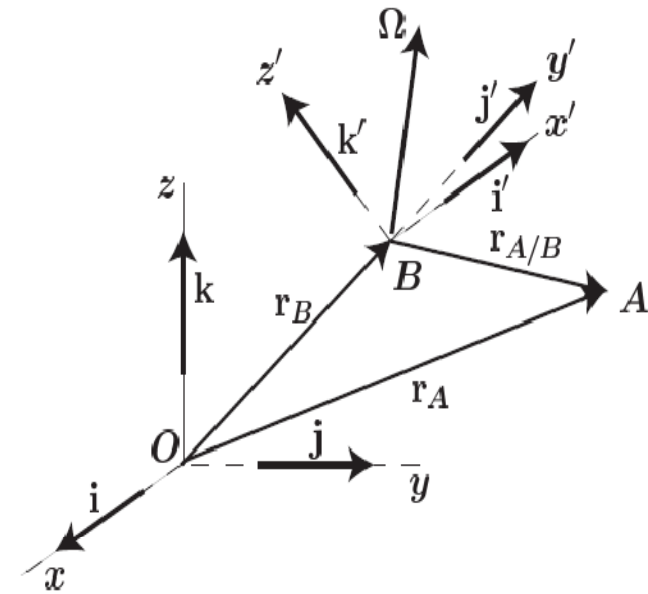
Differentiating velocity  $\mathbf{v}_A$  once again, and making use of Coriolis' theorem

Like,

$$d(\mathbf{v}_{A/B})_{x'y'z'}/dt = (\dot{\mathbf{v}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'}$$

$$\mathbf{V}_A = \mathbf{v}_B + (\mathbf{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times \mathbf{r}_{A/B}$$

$$\begin{aligned} \mathbf{a}_A = \dot{\mathbf{v}}_A &= \dot{\mathbf{v}}_B + (\dot{\mathbf{v}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} \\ &+ \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) \\ &= \mathbf{a}_B + (\mathbf{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) \end{aligned}$$



$$\begin{aligned}
\mathbf{a}_A = \dot{\mathbf{v}}_A &= \dot{\mathbf{v}}_B + (\dot{\mathbf{v}}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} \\
&+ \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) \\
&= \mathbf{a}_B + (\mathbf{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})
\end{aligned}$$

$\mathbf{a}_A$  and  $\mathbf{a}_B$  are the accelerations of A and B observed by xyz.

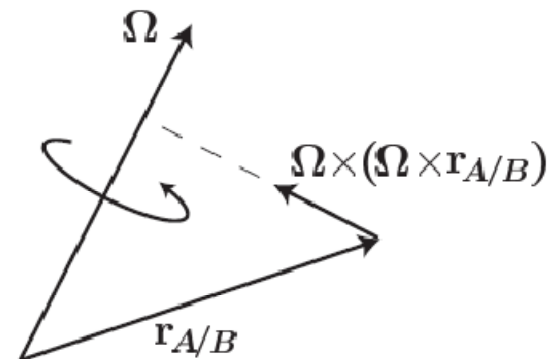
$(\mathbf{a}_{A/B})_{x'y'z'}$  is the acceleration of A measured by an observer B that rotates with the axes  $x'y'z'$ .

$2 \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'}$  -- Coriolis' acceleration

$\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B}$  is due to the change in  $\boldsymbol{\Omega}$  ---- Tangential acceleration

$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})$  is called centripetal acceleration

this acceleration always points towards the axis of rotation, and is orthogonal to  $\dot{\boldsymbol{\Omega}}$



angular acceleration vector  $\dot{\Omega}$  is the derivative taken with respect to the fixed observer, O, or with respect to the rotating observer, B.

The time derivatives of vectors which are parallel to  $\Omega$  are the same for both observers. This can be easily seen if we go back to Coriolis' theorem and apply it to  $\Omega$ . That is,

$$\dot{\Omega} = (\dot{\Omega})_{x'y'z'} + \Omega \times \Omega = (\dot{\Omega})_{x'y'z'}$$

### Non-rotational observers

for this case where  $\dot{\Omega} = \dot{\Omega} = 0$ , the  $x'y'z'$  axes only have translational motion relative to the fixed axes  $xyz$ , and therefore the equations reduce to

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

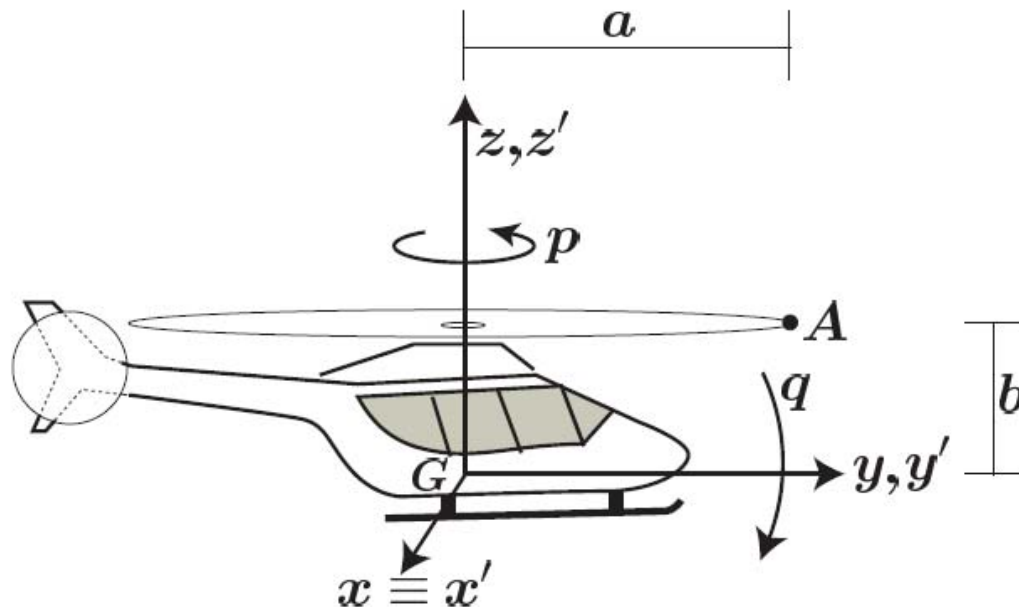
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

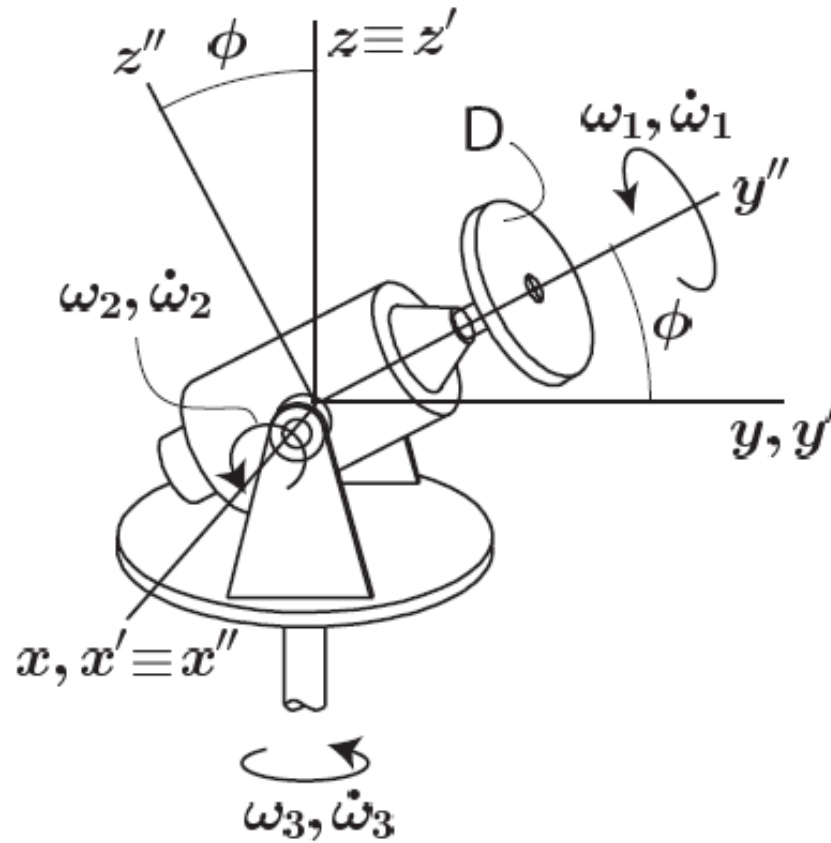


## Helicopter Blades motion problem

To determine the instantaneous velocity and acceleration of a point A, which is located at the tip of the blade of the helicopter. The blades are rotating with an angular velocity  $p$ , and, at the same time, the helicopter is pitching downwards at an angular velocity  $q$ . At the instant considered, the velocity and acceleration of the center of mass of the helicopter, G, are zero.



To determine the angular acceleration of the disc D as a function of the angular velocities and accelerations given in the diagram. The angle of  $\omega_1$  with the horizontal is  $\phi$ .



# Newton's Second Law for Non-Inertial Observers -- Inertial Forces

## Inertial reference frames

Expression that related the accelerations observed using two reference frames, A and B, which are in relative motion with respect to each other.

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) .$$

$\mathbf{a}_A$  is the acceleration of particle A observed by one observer

$(\mathbf{a}_{A/B})_{x'y'z'}$  is the acceleration of the same particle observed the other (moving) observer.

The acceleration of particle A will be different for each observer, unless all the other terms in the above expression are zero.

This means that if one of the observers is inertial, the other observer will be inertial if and only if  $\dot{\boldsymbol{\Omega}} = 0$ ,  $\boldsymbol{\Omega} \text{dot} = 0$  and  $\mathbf{a}_B = 0$ .

we conclude that,

- inertial frames can not rotate with respect to each other, i.e.,  $\dot{\Omega} = 0$ ,  $\ddot{\Omega} = 0$  and,
- inertial frames can not be accelerating with respect to each other, i.e.  $a_B = 0$ .

Thus, inertial frames can only be at most in constant relative velocity with respect to each other. In practical terms, the closest that we are able to get to an inertial frame is one which is in constant relative velocity with respect to the most distant stars.

## The earth as an inertial reference frame

Given that the earth is rotating about itself and at the same time is rotating about the sun, it is clear that the earth can not be an inertial reference frame. However, we shall see that, for many applications, the error made in assuming that the earth is an inertial reference frame is small

## Translating/Rotating observers

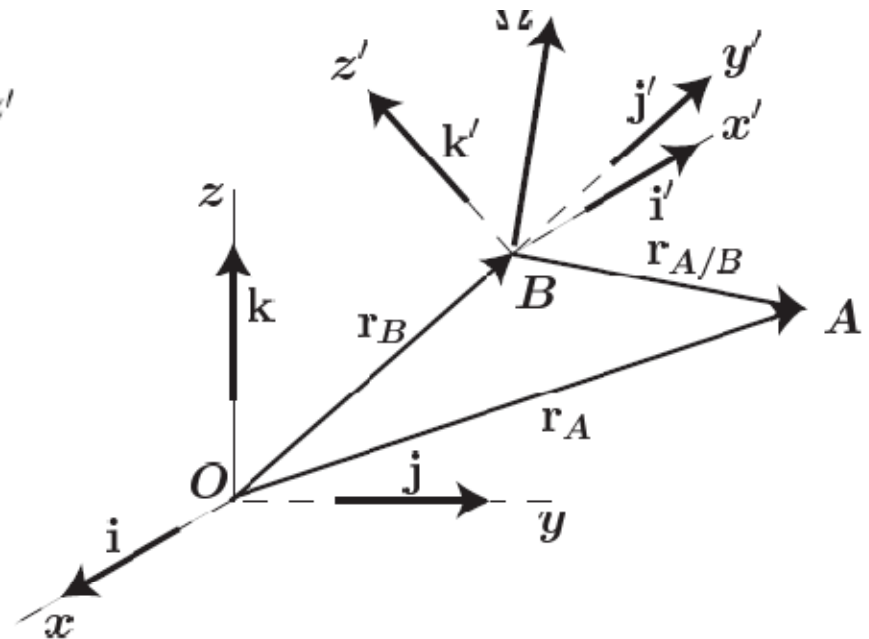
$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_{x'y'z'} + 2\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B})$$

$$\mathbf{F}' = m\mathbf{a}_A ,$$

$$\mathbf{F} - m\mathbf{a}_B - 2m\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = m(\mathbf{a}_{A/B})_{x'y'z'} .$$

$$\mathbf{F}_{inertial} = -m\mathbf{a}_B - 2m\boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m\dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}),$$

$$\mathbf{F}_{apparent} \equiv \mathbf{F} + \mathbf{F}_{inertial} = m(\mathbf{a}_{A/B})_{x'y'z'}$$



## Effect of earth's rotation.

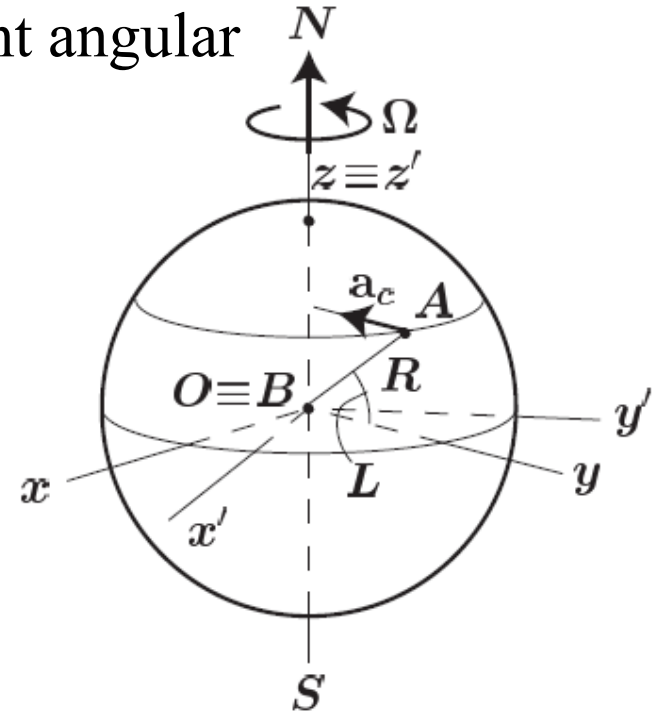
Consider for instance two reference frames  $xyz$  and  $x'y'z'$ . The first frame is fixed and the second frame rotates with the earth.

$\dot{\Omega} = 0$ , since the earth rotates with a constant angular velocity, and  $a_B = 0$ .

Centripetal acceleration,  $a_c = \dot{\Omega} \times \dot{\Omega} \times r_{A/B}$ , will depend on the point considered and is directed towards the axis of rotation.

modulus is given by  $a_c = R \Omega^2 \cos(\text{Lat})$ .

For the earth,  $\Omega = 7.3 \times 10^{-5} \text{ rad/s}$ ,  
 $R = 20 \times 10^6 \text{ ft}$ , and, if we consider, for instance,  
a point located at a latitude of  $L = 40^\circ$ , then,  
 $a_c = 0.08 \text{ ft/s}^2$ .



Coriolis acceleration,  $a_{cor} = 2 \Omega \times (v_{A/B})_{x'y'z'}$ , depends on the velocity of A relative to the rotating earth and is zero if the point is not moving relative to the earth. On the other hand, for an aircraft flying in the east–west direction, at a speed of 250 m/s ( 718 ft/s),  $a_{cor}$  would be in the radial direction at A (local vertical) and pointing away from the center of the earth (upwards). The magnitude will be  $a_{cor} = 0.10 \text{ ft/s}^2$ .

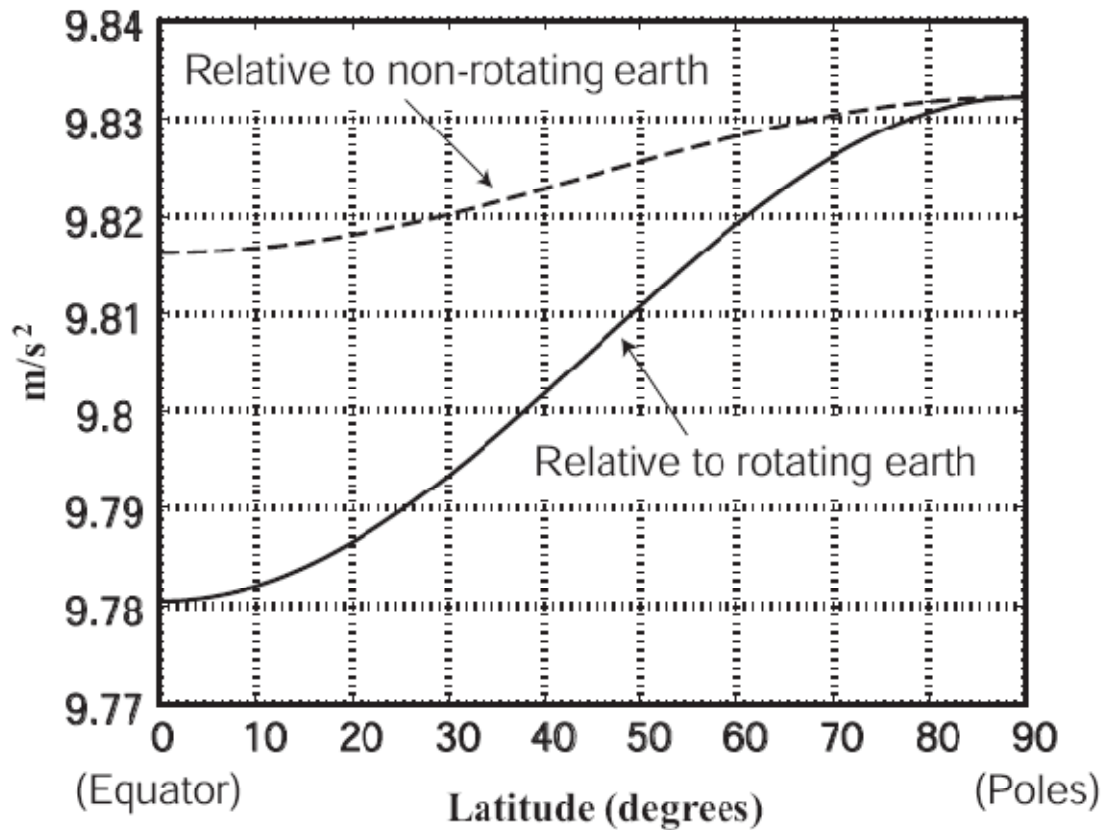
We see that these values, although not negligible in many situations, are still small when compared with the acceleration due to gravity of  $g = 32.2 \text{ ft/s}^2$ .

## Gravity variations due to Earth rotation

$$g \approx 9.780327(1 + 0.005279 \sin^2 L + 0.000024 \sin^4 L),$$

where  $L$  is the latitude of the point considered and  $g$  is given in  $\text{m/s}^2$ . The coefficient  $0.005279$  has two components:  $0.00344$ , due to Earth's rotation, and the rest is due to Earth's oblateness (or lack of sphericity).





gravitational acceleration at the poles is about 0.5% larger than at the equator. Furthermore, the deviations due to the Earth's rotation are about three times larger than the deviations due to the Earth's oblateness

## Gravity variations due to Earth rotation

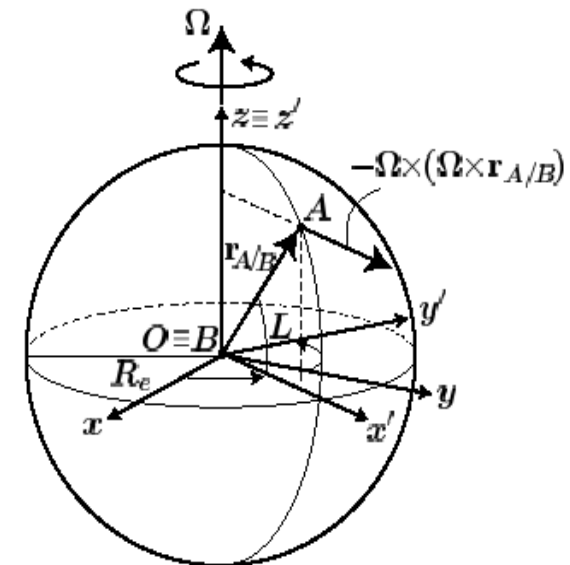
Influence of Earth's rotation on the gravity measured by an observer rotating with the Earth.

Consider two reference frames. A fixed frame  $xyz$ , and a frame  $x'y'z'$  that rotates with the Earth. Both the inertial observer,  $O$ , and the rotating observer,  $B$ , are situated at the center of the Earth, and are observing a mass  $m$  situated at point  $A$  on the Earth's surface.

$$F - m a_B - 2m \boldsymbol{\Omega} \times (\mathbf{v}_{A/B})_{x'y'z'} - m \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{A/B} - m \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{A/B}) = m (\mathbf{a}_{A/B})_{x'y'z'}$$

The forces on the mass will be the gravitational force,  $mg_0$ , and the reaction force,  $R$ , which is needed to keep the mass at rest relative to the Earth's surface

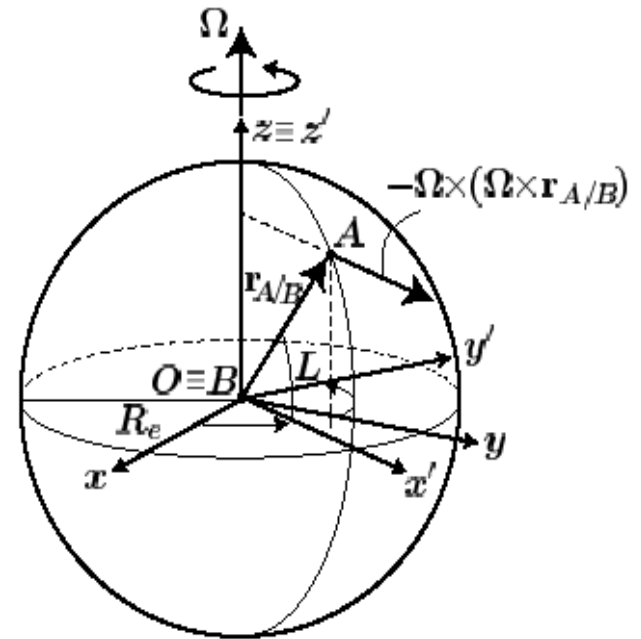
$$F = R + mg_0$$



Since the mass  $m$  is assumed to be at rest,  $\ddot{y} = 0$ , and,  $O = B$

$$R + mg_0 - m\Omega \times (\Omega \times r_{A/B}) = 0$$

$$R = -m[g_0 - \Omega \times (\Omega \times r_{A/B})] = -mg$$



An observer at rest on the surface of the Earth will observe a gravitational acceleration given by  $g = g_0 - \Omega \times (\Omega \times r_{A/B})$ . The term  $-\Omega \times (\Omega \times r_{A/B})$  has a magnitude  $\Omega^2 d = 2 \Omega^2 R_e \cos L$ , and is directed normal and away from the axis of rotation.

## Angular deviation of g

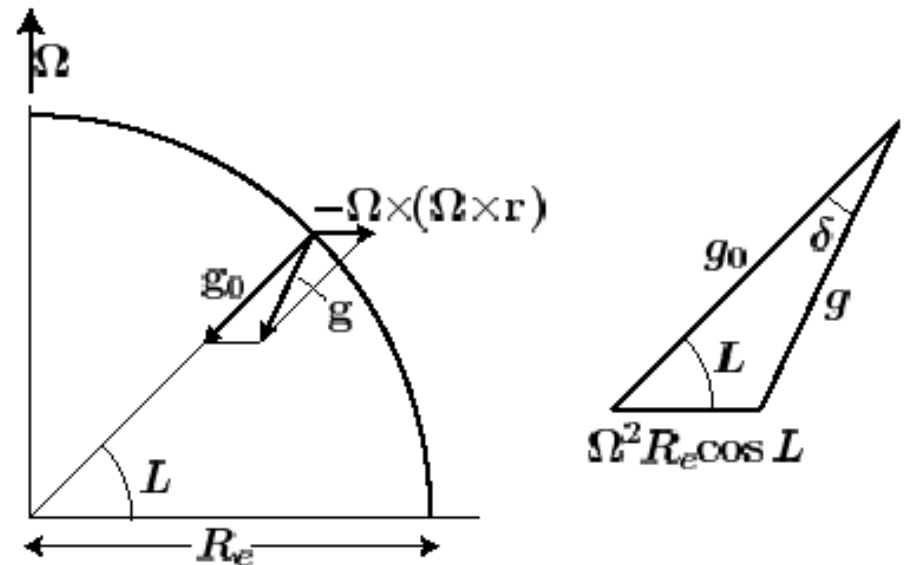
Consider a spherical Earth, and we want to determine the effect of Earth's rotation on the direction g.

An observer rotating with the Earth will observe a gravity vector

$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$$

where  $\mathbf{g}_0$  is the geocentric gravity, and  $\mathbf{g}$  is the modified gravity.

From the triangle formed by  $\mathbf{g}_0$ ,  $\mathbf{g}$ , and  $\boldsymbol{\Omega}^2 R \cos L$ , we have  $g \sin \delta = \boldsymbol{\Omega}^2 R \cos L \sin L = (\boldsymbol{\Omega}^2 R / 2) \sin 2L$ . We expect  $\delta$  to be small, and, therefore,  $\sin \delta \approx \delta$ , and  $g \approx g_0$ .



$$\delta \approx \frac{\Omega^2 R_e}{2g} \sin 2L .$$

which is maximum when  $L = \pm 45^\circ$ . In this case, we have  $\Omega = 7.29(10^{-5})$  rad/s,  $R_e = 6370$  km, and  $\max \delta = 1.7(10^{-3})$  rad  $\approx 0.1^\circ$ .