Rocket performance

$$T = -I_{sp} g_0 \frac{dm}{dt}$$

$$\frac{dm}{dt} = -\frac{T}{I_{sp} g_0}$$

Assuming the thrust and ISP constant

$$\Delta m = -\frac{T}{I_{sp} g_0} \Delta t$$

$$\Delta t = \frac{I_{sp} g_0}{T}(m_0 - m_f) = \frac{I_{sp} g_0}{T} m_0 \left(1 - \frac{m_f}{m_0}\right)$$

m_0 and m_f are the mass of the vehicle at the beginning and end of the burn,

mass ratio - ratio of the initial mass to final mass,

$$n = \frac{m_0}{m_f}$$

Mass ratio is always more than 1

$$\Delta t = \frac{n-1}{n} \frac{I_{sp}}{T/m_0 g_0}$$

T/mg_0 is the thrust-to-weight ratio.

Thrust-to-weight ratio for a launch vehicle at lift-off is typically in the range 1.3 to 2.

$$\frac{dv}{dt} = -I_{sp}g_0\frac{dm/dt}{m} - \frac{D}{m} - g\sin\gamma$$

Integrating the above from t_o to t_f

$$\Delta v = I_{sp}g_0\ln\frac{m_0}{m_f} - \Delta v_D - \Delta v_G$$

$$\Delta v_D = \int_{t_0}^{t_f}\frac{D}{m}dt \qquad \Delta v_G = \int_{t_0}^{t_f} g\sin\gamma\, dt$$

Sounding rocket of initial mass *m0 and mass mf after all propellant is consumed is* launched vertically (*γ =90◦). The propellant mass flow rate ˙m_e is constant. Neglecting* drag and the variation of gravity with altitude, calculate the maximum height *h* attained by the rocket. For what flow rate is the greatest altitude reached?

$$m = m_0 - \dot{m}_e t \qquad t_{bo} = \frac{m_0 - m_f}{\dot{m}_e}$$

$$\Delta v_G = \int_0^{t_{bo}} g_0 \sin(90^\circ) dt = g_0 t_{bo}$$

Let us define $I_{sp} g_0 = c$

Velocity as a function of time

$$v = c \ln \frac{m_0}{m_0 - \dot{m}_e t} - g_0 t$$

Since *dh/dt =v, the altitude as a function of time is*

$$h = \int_0^t v\,dt = \int_0^t \left(c \ln \frac{m_0}{m_0 - \dot{m}_e t} - g_0 t \right) dt$$

$$= \frac{c}{\dot{m}_e} \left[(m_0 - \dot{m}_e t) \ln \frac{m_0 - bt}{m_0} + \dot{m}_e t \right] - \frac{1}{2} g_0 t^2$$

$$t_{bo} = \frac{m_0 - m_f}{\dot{m}_e}$$

$$h_{bo} = \frac{c}{\dot{m}_e}\left(m_f \ln \frac{m_f}{m_0} + m_0 - m_f\right) - \frac{1}{2}\left(\frac{m_0 - m_f}{\dot{m}_e}\right)^2 g$$

$$v_{bo} = c \ln \frac{m_0}{m_f} - \frac{g_0}{\dot{m}_e}(m_0 - m_f)$$

After burnout, the rocket coasts upward with the constant downward acceleration of gravity

$$v = v_{bo} - g_0(t - t_{bo})$$

$$h = h_{bo} + v_{bo}(t - t_{bo}) - \frac{1}{2}g_0(t - t_{bo})^2$$

Substituting for v_{bo}, h_{bo} and t_{bo} in the above

$$v = c \ln \frac{m_0}{m_f} - g_0 t$$

$$h = \frac{c}{\dot{m}_e}\left(m_0 \ln \frac{m_f}{m_0} + m_0 - m_f\right) + ct \ln \frac{m_0}{m_f} - \frac{1}{2}g_0 t^2$$

maximum height h_{max} *is reached when v=0,*

$$c \ln \frac{m_0}{m_f} - g_0 t_{\max} = 0 \quad \Rightarrow \quad t_{\max} = \frac{c}{g_0} \ln \frac{m_0}{m_f}$$

Substituting t_{max} *in the following*

$$h = \frac{c}{\dot{m}_e}\left(m_0 \ln \frac{m_f}{m_0} + m_0 - m_f\right) + ct \ln \frac{m_0}{m_f} - \frac{1}{2} g_0 t^2$$

gives

$$\underline{h_{\max} = \frac{c m_0}{\dot{m}_e}(1 + \ln n - n) + \frac{1}{2}\frac{c^2}{g_0} \ln^2 n}$$

$$h_{\max} = \frac{cm_0}{\dot{m}_e}(1 + \ln n - n) + \frac{1}{2}\frac{c^2}{g_0}\ln^2 n$$

where *n is the mass ratio (n>1). Since n>(1+ ln n), it follows that (1+ ln n − n)* is negative.

Hence, *hmax can be increased by increasing the mass flow rate* $\dot{m}_e$

In fact, the greatest height is achieved when $\dot{m}_e \to \infty$*,*

i.e., all of the propellant is expended at once, like a mortar shell