ANALYTICAL FUNCTIONS

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Analytical functions :- (Regular functions or Holomorphic functions)

Definition:-

A Single valued function f(z) is said to be analytic at a point z_0 ,if it has a derivative at z_0 and at every point in some neighbourhood of z_0 .

Note :

If it is analytical at every point in a region R, then it is said to be analytic in the region R.

Necessary condition for a complex function $f(z)$ to be analytic:-

Derivation of Cauchy-Riemann equations:-

Statement:-

If $f(z) = u(x,y) + i\ v(x,y)$ is analytic in a region R of the z-plane then

i) u_x , u_y, v_x , v_y exist and

ii) $u_x = v_y$ and $u_y = -v_x$ at every point in that region.

Necessary condition for a complex function f(z) to be analytic:-

Derivation of Cauchy-Riemann equations:-

Proof:-

Let f(z) = u(x,y) + i v(x,y)

We first assume f(z) is analytic in a region R. Then by the definition, f(z) has a derivative f'(z) everywhere in R.

Now

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Derivation of Cauchy-Riemann equations:-

Let z = x+iy

$\Delta z = \Delta x + i\,\Delta y$

$\therefore (z+\Delta z) = (x+\Delta x) + i\,(y+\Delta y)$

$$\therefore f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$$

We know that, $f(z) = u(x,y) + i\,v(x,y)$

Now

$$f'(z) = \lim_{(\Delta x + i\Delta y)\to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + i\,v(x,y)]}{\Delta x + i\Delta y}$$

Derivation of Cauchy-Riemann equations:-

Case (i) :- If $\Delta z \rightarrow 0$, first we assume that $\Delta y=0$ and $\Delta x \rightarrow 0$

$\therefore$

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) + i\,v(x+\Delta x, y)] - [\,u(x,y) + \,i\,v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - \,u(x,y)] + i\,[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - \,u(x,y)]}{\Delta x} + \,i\, \lim_{\Delta x \to 0} \frac{[v(x+\Delta x, y) - \,v(x,y)]}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

$\therefore$ $f'(z) = u_x + i\, v_x$ -------→(1)

Derivation of Cauchy-Riemann equations:-

Case (ii) :- If $\Delta z \to 0$, now we assume $\Delta x = 0$ and $\Delta y \to 0$

$\therefore$

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x,y+\Delta y) + i\,v(x,y+\Delta y)] - [\,u(x,y) + \,i\,v(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[u(x,y+\Delta y) - \,u(x,y)] + i\,[v(x,y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{[u(x,y+\Delta y) - \,u(x,y)]}{i\Delta y} + \,i\,\lim_{\Delta y \to 0} \frac{[v(x,y+\Delta y) - \,v(x,y)]}{i\Delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$\therefore$ $f'(z) = -i\,u_y + v_y$ -------→(2) (since $1/i = -i$)

Derivation of Cauchy-Riemann equations:-

from (1) & (2) , we get

$$u_x + i\, v_x = -i\, u_y + v_y$$

Equating real and imaginary parts weget,

$$u_x = v_y \quad \text{and} \quad u_y = -\, v_x$$

The above equations are called Cauchy-Riemann equations (or) C-R Equations .

Therefore the function f(z) to be analytic at the point z, it is necessary that the four partial derivatives u_x , u_y, v_x, v_y should exist and satisfy the C-R equations.

Sufficient condition for f(z) to be analytic

Statement:- The singled valued continuous function $f(z) = u + i v$ is analytic in a region R of the z-plane, if the four partial derivatives u_x , u_y , v_x , v_y , (i) exist , (ii) continuous , (iii) they satisfy the C-R equations $u_x = v_y$ and $u_y = - v_x$ at every point of R.

Note:- All polynomials, trigonometric, exponential functions are continuous.

Cauchy-Riemann Equations in Polar form

Statement:- If f(z) = u(r,θ) + i v(r,θ) is differential ar $z = re^{i\theta}$, then

$$\frac{\partial u}{\partial r} = \left(\frac{1}{r}\right)\frac{\partial v}{\partial \theta} \Rightarrow u_r = \left(\frac{1}{r}\right)v_\theta$$

$$\frac{\partial v}{\partial r} = -\left(\frac{1}{r}\right)\frac{\partial u}{\partial \theta} \Rightarrow v_r = -\left(\frac{1}{r}\right)u_\theta$$

Proof:- Let $z = re^{i\theta}$

and f(z) = u+iv

i.e., $u+iv = f(re^{i\theta})$ → (1)

Cauchy-Riemann Equations in Polar form

Differentiating partially w.r.t. \`r´ weget,

$$\frac{\partial u}{\partial r}+i\,\frac{\partial v}{\partial r}= f'(re^{i\theta})e^{i\theta} \longrightarrow \quad (2)$$

Differentiating partially w.r.t. \`θ´ weget,

$$\frac{\partial u}{\partial \theta}+i\,\frac{\partial v}{\partial \theta}= f'(re^{i\theta})(re^{i\theta})(i)$$

$$=(ri)\,f'(re^{i\theta})(e^{i\theta})$$

$$=(ri)\left[\frac{\partial u}{\partial r}+i\,\frac{\partial v}{\partial r}\right] \quad (\mathit{from\ eqn.}\ (2))$$

$$= ir\left(\frac{\partial u}{\partial r}\right)-r\left(\frac{\partial v}{\partial r}\right) \quad \rightarrow (3)$$

Cauchy-Riemann Equations in Polar form

Equating real and imaginary parts in eqn. (3) , weget,

$$\frac{\partial u}{\partial \theta} = -r\,\frac{\partial v}{\partial r} \qquad and \qquad \frac{\partial v}{\partial \theta} = r\,\frac{\partial u}{\partial r}$$

$$i.e., u_\theta = -r\, v_r \ \ and \quad v_\theta = r\, u_r$$

$$(or) \quad \boxed{v_r = \left(\frac{-1}{r}\right) u_\theta \ \ and \quad u_r = \left(\frac{1}{r}\right) v_\theta}$$

EXAMPLES

1) Show that $f(z) = z^3$ is analytic.

Proof:- Given $f(z) = z^3 = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

We know that $f(z) = u+iv$

So ,

$$u = x^3 - 3xy^2 \quad , \quad v = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad , \quad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy \quad , \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

EXAMPLES

$$\text{from the above equations we get,}$$

$$u_x = v_y \quad and \quad u_y = -v_x$$

$$\therefore\ C-R \text{ equations are satisfied.}$$

Here u_x , u_y , v_x , v_y exists and continuous.

Hence the given function f(z) is analytic.

2) Examine the analyticity of the following functions and find its derivatives.

$$i)\quad f(z) = e^z$$

$$ii)\quad f(z) = \cos z$$

$$iii)\quad f(z) = \sinh z$$

EXAMPLES

i) Solution:-

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$

$$\text{Here} \quad u = e^x \cos y \quad \text{and} \quad v = e^x \sin y$$

$$u_x = e^x \cos y \qquad v_x = e^x \sin y$$

$$u_y = -e^x \sin y \qquad v_y = e^x \cos y$$

$$\therefore \quad u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$$\Rightarrow \quad C-R \text{ equations are satisfied.}$$

$$\Rightarrow \quad f(z) \text{ is analytic everywhere in the complex plane.}$$

EXAMPLES

$$Now\ \ f'(z) = u_x + i\ v_x$$

$$= e^x \cos y \ \ + i\ e^x \sin y$$

$$= e^x \ (\cos y + i \sin y)$$

$$= e^x \ e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$

EXAMPLES

ii) Solution:-

$$f(z) = \cos z$$
$$= \cos(x+iy)$$
$$= \cos x \cos(iy) - \sin x \sin(iy)$$
$$= \cos x \cosh y - i \sin x \sinh y \quad (\because \cos(ix) = \cosh x$$
$$\sin(ix) = i \sinh x)$$

$$\therefore \quad u = \cos x \cosh y \qquad v = -\sin x \sinh y$$
$$u_x = -\sin x \cosh y \qquad v_x = -\cos x \sinh y$$
$$u_y = \cos x \sinh y \qquad v_y = -\sin x \cosh y$$
$$\Rightarrow u_x = v_y \quad and \quad u_y = -v_x$$

EXAMPLES

$\therefore\ C-R \text{ equations satisfied}$

$\Rightarrow\ \text{It is analytic}$

$$\text{Also} \quad f'(z) = u_x + i\, v_x$$

$$= \left(-\sin x \cosh y\right) + i\left(-\cos x \sinh y\right)$$

$$= -\sin x \cos iy + i(-\cos x \left(\frac{1}{i}\right)\sin(iy))$$

$$= -\sin x \cos\left(iy\right) - \cos x \sin(iy)$$

$$= -\left[\sin(x+iy)\right]$$

$$= -\sin z$$

EXAMPLES

iii) Solution:-

$$f(z) = \sinh z = \frac{1}{i} \sin(iz)$$

$$= -i(\sin i(x+iy))$$

$$= -i(\sin(ix)\cos y - \cos(ix)\sin y)$$

$$= -i(i\sinh x \cos y - \cosh x \sin y)$$

$$= \sinh x \cos y + i \cosh x \sin y$$

$$\therefore \quad u = \sinh x \cos y \quad , \quad v = \cosh x \sin y$$

$$u_x = \cosh x \cos y \quad , \quad v_x = \sinh x \sin y$$

$$u_y = -\sinh x \sin y \quad , \quad v_y = \cosh x \cos y$$

$$\Rightarrow \quad u_x = v_y \quad and \quad u_y = -v_x$$

EXAMPLES

$$\therefore \quad C - R \ equations \ are \ satisfied$$

$$\Rightarrow \quad f(z) \ is \ analytic.$$

$$Now \qquad f'(z) = u_x + i\, v_x$$

$$= (\cosh x \cos y) + i\,(\sinh x \sin y)$$

$$= (\cos(ix) \cos y) + i\left(\left(\frac{1}{i}\right)\sin(ix) \sin y\right)$$

$$= \cos(ix - y)$$

$$= \cos i\,(x + iy) \quad \left(\because\ (1/i) = -i\right)$$

$$= \cos i\,z$$

$$= \cosh z$$

TRY IT

Examine the analyticity of the following functions and find its derivatives.

$i)\quad f(z) = e^{x}(\cos y + i \sin y)$

$ii)\quad f(z) = e^{-x}(\cos y - i \sin y)$

$iii)\quad f(z) = \sin x \cosh y + i \cos x \sinh y$

EXAMPLES

3) $Show\ that\ the\ function\ f(z)=\sqrt{|xy|}\ is\ not\ regular\ (analytic)$
$at\ the\ origin,\ although\ C-R\ equations\ are\ satisfied$
$at\ the\ origin.$

$Solution:-$

$$Given\ \ f(z)=\sqrt{|xy|}$$

$$Hence\ \ u=\sqrt{|xy|}\quad and\quad v=0$$

$$Now,\ u_x=\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}$$

EXAMPLES

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = 0$$

llly $u_y(0,0) = 0$

$v_x(0,0) = 0$

$v_y(0,0) = 0$

$\Rightarrow \; u_x = v_y$ *and* $u_y = -v_x$ *at the origin.*

$\therefore$ $C - R$ *equations are satisfied at the origin.*

EXAMPLES

But $$f'(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{(\Delta x + i\Delta y) \to 0} \frac{\sqrt{|\Delta x \Delta y|} - 0}{\Delta z}$$

Along the curve y = mx

$$f'(0) = \lim_{\substack{\Delta y = m\Delta x \\ \Delta x \to 0}} \frac{\sqrt{|m||\Delta x|^2}}{\Delta x(1+im)} = \frac{\sqrt{|m|}}{1+im}$$

$\therefore$ *The* limit is not unique, since it depends on 'm'.

$\therefore$ f′(0) does not exist.

Hence f(z) is not regular at the origin.

C-R equations in polar form
EXAMPLES

1) Check for analyticity of logz

(or) Show that f(z) = logz is analytic everywhere except at the origin and find its derivatives.

Solution:-

$$f(z) = \log z$$

$$= \log(r e^{i\theta}) \qquad (\because z = r e^{i\theta})$$

$$= \log r + \log e^{i\theta}$$

$$= \log r + i\theta$$

$$\underline{w.k.t.} \quad f(z) = u + iv$$

$$Here \quad u = \log r \quad and \quad v = \theta$$

EXAMPLES

$$\therefore \qquad u_r = \frac{1}{r} \qquad\qquad v_r = 0$$

$$u_\theta = 0 \qquad\qquad v_\theta = 1$$

$\therefore \qquad u_r, u_\theta, v_r, v_\theta$ exist, are continuous and satisfy C-R equations

$\mathrm{u}_r = \left(\frac{1}{r}\right)\mathrm{v}_\theta$ and $\mathrm{v}_r = -\left(\frac{1}{r}\right)\mathrm{u}_\theta$ *everywhere except at* $r = 0 \ (i.e.)\ z = 0.$

$\therefore \qquad f(z)$ *is analytic everywhere except at* $z = 0.$

EXAMPLES

2) Prove that f(z) = z^n is analytic function and find its derivatives.

Proof:-

$$f(z) = z^n = (re^{i\theta})^n$$
$$= r^n \; e^{in\theta}$$
$$= r^n \left[\cos n\theta + i \sin n\theta\right]$$

$$\therefore \quad u = r^n \cos n\theta \quad ; \quad v = r^n \cos n\theta$$
$$u_r = nr^{n-1} \cos n\theta \quad ; \quad v_r = nr^{n-1} \sin n\theta$$
$$u\theta = -nr^n \sin n\theta \quad ; \quad v_\theta = nr^n \cos n\theta$$
$$\Rightarrow \quad u_r = \left(\frac{1}{r}\right) v_\theta \quad and \quad v_r = -\left(\frac{1}{r}\right) u_\theta$$

EXAMPLES

$Thus\ u_r, u_\theta, v_r, v_\theta$ *exist, are continuous and satisfy* $C - R$ *equations everywhere.*

$\therefore\ f(z)$ *is analytic.*

Also $$f'(z) = \left(\frac{u_r + i v_r}{e^{i\theta}}\right)$$

$$= \frac{\left(nr^{n-1}\cos n\theta\right) + i\left(nr^{n-1}\sin n\theta\right)}{e^{i\theta}}$$

$$= \frac{nr^{n-1}\left[\cos n\theta + i\sin n\theta\right]}{e^{i\theta}}$$

$$= \frac{nr^{n-1}e^{in\theta}}{e^{i\theta}} = n\left(re^{i\theta}\right)^{n-1} = nz^{n-1}$$

Laplace Equations

In Cartesian form:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$i.e., \nabla^2 \phi = 0$$

In Polar form:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

HARMONIC FUNCTIONS

A real valued function of two real variables x and y is said to be harmonic, if

i) The second order partial derivatives u_{xx} , u_{xy} , u_{yx} , u_{yy} exist and they are continuous.

and

ii) The Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ satisfies.

Conjugate Harmonic functions:-

If u+iv is an analytic function of z then v is called a conjugate harmonic function of u; (or) u is called a conjugate harmonic function of v; (or) u and v are called conjugate harmonic functions.

Properties of Analytic functions

Property (1) :- The real and imaginary parts of an analytic function $f(z) = u+iv$ satisfy the Laplace equation (or) real part "u" and imaginary part "v" of an analytic function $f(z) = u+iv$ are harmonic functions.

Proof:-

Given $f(z) = u+i\,v$ is an analytic function.

i.e., u and v are continuous, u_x , u_y , v_x , v_y are exist and they satisfy the C-R equations $u_x = v_y$ and $u_y = -v_x$

$\longrightarrow$ (1) $\longrightarrow$ (2)

Properties of Analytic functions

Diff. eqn.(1) partially w.r.t. x, we get,

$$u_{xx} = v_{yx} \rightarrow (3)$$

Diff. eqn.(2) partially w.r.t. y, we get,

$$u_{yy} = -v_{xy} \rightarrow (4)$$

Adding (3) & (4) we get,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \qquad [\text{Q } v_{yx} = v_{xy}]$$

$\therefore$ *u satisfies Laplace equation.*

Hence u is a Harmonic function.

Properties of Analytic functions

Now ,

$\underline{Diff.\ eqn.(1)\ partially\ w.r.t.\ y,\ we\ get,}$

$$u_{xy} = v_{yy} \rightarrow (5)$$

$\underline{Diff.\ eqn.(2)\ partially\ w.r.t.\ x,\ we\ get,}$

$$u_{yx} = -v_{xx} \rightarrow (6)$$

$\underline{subracting\ (5)\ \&\ (6)\ we\ get,}$

$$v_{yy} + v_{xx} = u_{xy} - u_{yx} = 0 \qquad \left[\because u_{xy} = u_{yx}\right]$$

$\therefore\ v\ satisfies\ Laplace\ equation.$

$Hence\ v\ is\ a\ Harmonic\ function.$

$Thus\ u\ and\ v\ are\ harmonic\ functions.$

Note:- The converse of the above result need not be true.

Properties of Analytic functions

Try it

Prove that the real and imaginary parts of an analytic function f(z) = u(r,θ) + i v(r, θ) satisfy the Laplace equation in polar coordinates.

$$i.e.,\ To\ prove\ that$$

$$u_{rr} + \left(\frac{1}{r}\right)u_r + \left(\frac{1}{r^2}\right)u_{\theta\theta}$$

$$and$$

$$v_{rr} + \left(\frac{1}{r}\right)v_r + \left(\frac{1}{r^2}\right)v_{\theta\theta}$$

Properties of Analytic functions

Orthogonal Curves:-

Two curves are said to be orthogonal to each other then they intersect at right angles. [product of slopes $m_1 m_2 = -1$]

Property (2) :-

If f(z) = u+ iv is an analytic function then the family of curves u(x,y) = a and v(x,y) = b (where a&b are constants) cut each other orthogonally.

Proof:-

$$\underline{Given:} \, u(x, y) = a \;\; and \;\; v(x, y) = b$$

Taking differentials on both sides, weget,

$$du = 0$$

Properties of Analytic functions

$$\Rightarrow \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{-u_x}{u_y} = m_1}$$

$$Illy \quad v(x, y) = b$$

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} = 0$$

$$\therefore \quad \boxed{\frac{dy}{dx} = \frac{-v_x}{v_y} = m_2}$$

Properties of Analytic functions

$$\text{Product of slopes, } m_1 \; m_2 = \left(\frac{-u_x}{u_y} \right) \left(\frac{-v_x}{v_y} \right)$$

$$= \frac{(-u_x)(u_y)}{(u_y)(u_x)} \qquad \begin{bmatrix} \because u_x = v_y \\ \text{and } u_y = -v_x \end{bmatrix}$$

$$= -1$$

Hence the two curves in eqns. (3) & (4) are orthogonal curves.

Properties of Analytic functions

Result :- (1) An analytic function with constant modulus is constant.

Proof:-

Let f(z) = u + i v be an analytic function

$$\therefore |f(z)| = \sqrt{u^2 + v^2}$$

$\underline{Given}:$ $|f(z)| = c$

$$i.e., \sqrt{u^2 + v^2} = c$$

$$\Rightarrow u^2 + v^2 = c^2 \rightarrow (1)$$

Properties of Analytic functions

$\underline{Diff.\ eqn.(1)\ partially\ w.r.t.\ x,\ we\,get,}$

$$2uu_x + 2vv_x = 0 \quad \Rightarrow \boxed{uu_x + vv_x = 0} \rightarrow (2)$$

$\underline{Diff.\ eqn.(1)\ partially\ w.r.t.\ y,\ we\,get,}$

$$2uu_y + 2vv_y = 0 \quad \Rightarrow \boxed{uu_y + vv_y = 0} \rightarrow (3)$$

$Since\ \ f(z)\ is\ analytic,\ it\ satisfies\ C-R\ equations$

$i.e.,\ u_x = v_y \quad and \quad u_y = -v_x$

$$\therefore \quad (2) \Rightarrow \ uu_x + v(-u_y) = 0 \quad \Rightarrow \ uu_x - vu_y = 0$$

$$(3) \Rightarrow \ uu_y + v(u_x) = 0 \quad \Rightarrow \ uu_y + vu_x = 0$$

Properties of Analytic functions

Squaring and adding the above equations, weget,

$$\left(uu_x - vu_y\right)^2 + \left(uu_y + vu_x\right)^2 = 0$$

$$\Rightarrow \; u^2u_x^2 + v^2u_y^2 - 2uvu_xu_y + u^2u_y^2 + v^2u_x^2 + 2uvu_yu_x = 0$$

$$\Rightarrow \; u^2\left[u_x^2 + u_y^2\right] + v^2\left[u_x^2 + u_y^2\right] = 0$$

$$\Rightarrow \; \left(u^2 + v^2\right)\left(u_x^2 + u_y^2\right) = 0$$

$$But \;\; u^2 + v^2 = c^2 \neq 0 \quad (\,from\; eqn.\,(1))$$

$$\therefore \;\; \boxed{u_x^2 + u_y^2 = 0} \;\; \rightarrow (4)$$

Properties of Analytic functions

Since

$$f(z) = u + iv$$

$$f'(z) = u_x + i\, v_x$$

$$= u_x - i\, u_y \quad (\, by\ C - R\ eqns.)$$

$$\therefore \quad |f'(z)| = \sqrt{u_x^2 + u_y^2}$$

$$\Rightarrow |f'(z)|^2 = u_x^2 + u_y^2$$

$$= 0 \quad (\, from\ (4))$$

$$\Rightarrow \quad f'(z) = 0$$

$$\Rightarrow \quad f(z)\ is\ a\ \text{constant}$$

$$\therefore \quad An\ analytic\ \ function\ with\ \text{constant modulus is constant.}$$

Properties of Analytic functions

Result :- (2) If f(z) = u+iv is a regular function of z = x+iy then

$$\nabla^2 \left[|f(z)| \right]^2 = 4\,|f'(z)|^2$$

Proof :–

To prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4\,|f'(z)|^2$

Let $f(z) = u + iv$

$$\overline{f(z)} = u - iv$$

$$\therefore \quad f(z)\,\overline{f(z)} = (u+iv)(u-iv) = u^2 + v^2$$

$$\therefore \quad \boxed{|f(z)|^2 = u^2 + v^2}$$

Properties of Analytic functions

Now,

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^2+v^2)$$

$$= \frac{\partial^2}{\partial x^2}(u^2)+\frac{\partial^2}{\partial x^2}(v^2)+\frac{\partial^2}{\partial y^2}(u^2)+\frac{\partial^2}{\partial y^2}(v^2) \rightarrow (1)$$

Now, consider, $\frac{\partial}{\partial x}(u^2) = 2uu_x$

$$\therefore \quad \frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x) = 2uu_{xx} + 2u_x^2$$

llly $\quad \frac{\partial^2}{\partial y^2}(u^2) = 2uu_{yy} + 2u_y^2$

Properties of Analytic functions

$$\therefore \quad \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} = 2u\left(u_{xx} + u_{yy}\right) + 2\left(u_x^2 + u_y^2\right)$$

$$= 2\left[u(0) + u_x^2 + u_y^2\right] \quad \left[\begin{array}{r} \because \ f(z) \text{ is analytic} \\ u \text{ is harmonic} \end{array}\right]$$

$$= 2\left[u_x^2 + (-v_x)^2\right] \quad \left[\begin{array}{l} \because \ f(z) \text{ is analytic,} \\ \Rightarrow C - R \text{ eqns. satisfied} \end{array}\right]$$

$$= 2\left[u_x^2 + v_x^2\right]$$

$$= 2\left|f'(z)\right|^2 \qquad \left(\begin{array}{l} \because \ f'(z) = u_x + i\,v_x \\ \Rightarrow \left|f'(z)\right| = \sqrt{u_x^2 + v_x^2} \end{array}\right)$$

Properties of Analytic functions

$$lllly \qquad \frac{\partial^2 v^2}{\partial x^2} + \frac{\partial^2 v^2}{\partial y^2} = 2\,|\,f'(z)|^2$$

$$\therefore \quad (1) \Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|\,f(z)|^2 = 2\,|\,f'(z)|^2 + 2\,|\,f'(z)|^2$$

$$= 4\,|\,f'(z)|^2$$

Thus proved

EXAMPLES

1) *If* $f(z) = e^{x}(\cos y + i \sin y)$ *is analytic function prove that* u, v *are harmonic functions.*

Solutions :–

To prove that u *and* v *are Harmonic functions*

i.e., T.P.T. $u_{xx} + u_{yy} = 0$ *and* $v_{xx} + v_{yy} = 0$

Here

$$u = e^{x} \cos y \qquad v = e^{x} \sin y$$

$$u_{x} = e^{x} \cos y \qquad v_{x} = e^{x} \sin y$$

$$u_{xx} = e^{x} \cos y \qquad v_{xx} = e^{x} \sin y$$

$$u_{y} = -e^{x} \sin y \qquad v_{y} = e^{x} \cos y$$

$$u_{yy} = -e^{x} \cos y \qquad v_{yy} = -e^{x} \sin y$$

EXAMPLES

$$\therefore \quad u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$$

$$and \quad v_{xx} + v_{yy} = e^x \sin y - e^x \sin y = 0$$

$\therefore$ *Both u & v satisfies Laplace equation*

Hence u & v are Harmonic functions.

CONSTRUCTION OF ANALYTIC FUNCTION

Milne-Thomson method :-

To find the analytic function f(z):

i) when u(x , y) is given (i.e., real part is given)

$$f(z) = \int u_x(z,0)\,dz - i \int u_y(z,0)\,dz$$

ii) when v(x ,y) is given (i.e., Imaginary part is given)

$$f(z) = \int v_y(z,0)\,dz + i \int v_x(z,0)\,dz$$

CONSTRUCTION OF ANALYTIC FUNCTION

Method to find out the Harmonic conjugate:

Let f(z) = u + i v be an analytic function.

Given: u(x ,y)

$$\boxed{\therefore \quad v = \int -u_y \, dx + \int u_x \, dy}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} \textit{treating } y \\ \textit{as constant} \end{pmatrix} \begin{pmatrix} \textit{Integrating the terms} \\ \textit{independent of } x \end{pmatrix}$$

EXAMPLES

1) If $u(x,y) = x^2 + y^2$, find $v(x,y)$ and Hence find $f(z)$.

Solution: –

Given: $u = x^2 - y^2$

$$\Rightarrow \quad u_x = 2x \quad , \quad u_y = -2y$$

we know that,

$$\boxed{v = \int -u_y \, dx + \int u_x \, dy}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} \text{treating } y \\ \text{as constant} \end{pmatrix} \begin{pmatrix} \text{Integrating the terms} \\ \text{independent of } x \end{pmatrix}$$

EXAMPLES

$$\therefore \quad v = \int -(-2y)\,dx + \int 2x\,dy$$

$$= 2xy + 0 \quad \left(\begin{array}{l}\text{IInd integral is zero since} \\ \text{there is no term indep. of "x"}\end{array}\right)$$

$$\Rightarrow \quad v = 2xy$$

$$\therefore \quad f(z) = u + i\,v$$

$$\Rightarrow f(z) = \left(x^2 - y^2\right) + i\left(2xy\right)$$

$$= x^2 + i^2 y^2 + 2x(iy)$$

$$= \left(x + i\,y\right)^2$$

$$\boxed{\therefore \quad f(z) = z^2}$$

EXAMPLES

1) Find f(z), when u(x ,y) = x^2 + y^2 .

(same example, using Milne-Thomson method, finding f(z))

Solution: –

Given: $u = x^2 - y^2$

$\Rightarrow \quad u_x = 2x \;, \qquad u_y = -2y$

$\therefore \quad u_x(z,0) = 2z \;,\; u_y(z,0) = 0$

By Milne-Thomson method ,

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz$$

$$= \int 2z\,dz - i\int 0\,dz$$

$$\therefore \quad \boxed{f(z) = z^2}$$

EXAMPLES

2) *Show that the function u(x ,y) = sinx coshy is harmonic.*

Find its harmonic conjugate v(x , y) and the analytic function

f(z) =u + i v .

$\underline{Solution:-}$

$\underline{Given}: \quad u = \sin x \cosh y$

$u_x = \cos x \cosh y \qquad u_y = \sin x \sinh y$

$u_{xx} = -\sin x \cosh y \qquad u_{yy} = \sin x \cosh y$

$\therefore \quad u_{xx} + u_{yy} = 0$

$\Rightarrow \quad u \text{ is harmonic.}$

EXAMPLES

$\underline{To\ \ find\ \ v(x,y)}:-$

$$we\ know\ that,\ v = \int -u_y\,dx + \int u_x\,dy$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix} treating\ \ y \\ as\ \text{constant} \end{pmatrix} \begin{pmatrix} Integrating\ the\ terms \\ independent\ \ of\ \ x \end{pmatrix}$$

$$\therefore\ V = \int -(\sin x\ \sinh y)\,dx + \int (\cos x\ \cosh y)\,dy$$

$$= -\sinh y \int \sin x\ \,dx + 0 \qquad \begin{bmatrix} \text{since } no\ term\ is \\ independent\ \ of\ \ x \end{bmatrix}$$

$$= -\sinh y\ (-\cos x)$$

$$\therefore \quad \boxed{V = \cos x\ \sinh y}$$

EXAMPLES

Now,

$$f(z) = u + iv = \sin x \cosh y + i \cos x \sinh y$$

$$= \sin x \cos(iy) + i \cos x \left(\frac{\sin(iy)}{i} \right)$$

$$= \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin(x + iy)$$

$$= \sin z$$

$$\therefore \quad \boxed{f(z) = \sin z}$$

EXAMPLES

3) Construct analytic function f(z) of which imaginary part

$v(x,y) = -2\sin x\,(e^{y} - e^{-y})$.

Solution :−

Given : $v(x,y) = -2\sin x\,(e^{y} - e^{-y})$

$$i.e.,\; v = -4\sin x \sinh y \qquad [\because e^{y} - e^{-y} = 2\sinh y]$$

$$v_x = -4\cos x \sinh y \quad , \quad v_y = -4\sin x \cosh y$$

$$\therefore \quad v_x(z,0) = 0 \quad , \quad v_y(z,0) = -4\sin z$$

$$\therefore \quad f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz$$

$$= \int -4\sin z\,dz$$

$$\Rightarrow \quad \boxed{f(z) = 4\cos z + c}$$

EXAMPLES

4) Find the analytic function f(z) = u+iv such that,

$u+v = x^3 + 3x^2 y - 3xy^2 - y^2 + 4x + 5$ and $f(0) = 2+3i$.

Solution :–

$$we\,know\,that, \quad f(z) = u + iv$$

$$i\ f(z) = iu - v$$

$$\therefore\ f(z) + i\, f(z) = u + iv + iu - v$$

$$\Rightarrow\ f(z)\,(1+i) = (u - v) + i(u + v)$$

$$F(z) = U + i\,V$$

$$where\ F(z) = f(z)(1+i)$$

$$U = (u - v),\ V = u + v = x^3 + 3x^2y - 3xy^2 - y^2 + 4x + 5$$

EXAMPLES

By Milne-thomson method,

$$F(z) = \int v_y(z,0)\, dz + i \int v_x(z,0)\, dz$$

$$Now, \quad v_x = 3x^2 + 6xy - 3y^2 + 4$$

$$v_y = 3x^2 - 6xy - 2y$$

$$v_x(z,0) = 3z^2 + 4$$

$$v_y(z,0) = 3z^2$$

$$\therefore \quad F(z) = \int 3z^2\, dz + i \int \left(3z^2 + 4\right) dz$$

$$= \frac{3z^3}{3} + i\left(\frac{3z^3}{3} + 4z\right)$$

EXAMPLES

$$\therefore \quad F(z) = z^3 + i\,(z^3 + 4) + c$$

$$\therefore \quad (1+i)\; f(z) = z^3(1+i) + i4z + c$$

$$\therefore \quad f(z) = z^3 + \frac{i4z}{(1+i)} + \frac{c}{(1+i)}$$

$$= z^3 + \frac{i4z\,(1-i)}{(1+i)(1-i)} + c_1$$

$$= z^3 + \frac{4z(i+1)}{2} + c_1$$

$$\therefore \quad f(z) = z^3 + 2z(1+i) + c_1 \quad \rightarrow (1)$$

EXAMPLES

$\underline{Given}:\ f(0) = 2+3i$

$put\ \ z=0\ in\ (1),\ weget,\ f(0) = c_1$

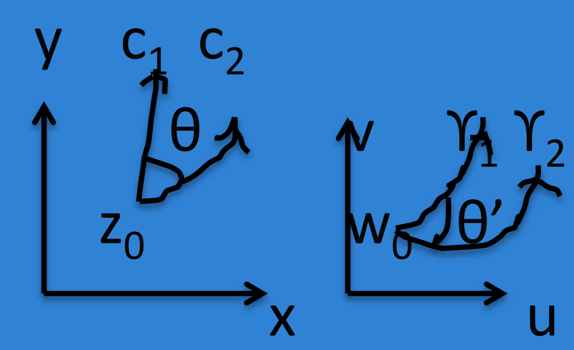
$$\therefore \quad \boxed{c_1 = 2+3i}$$

$$\therefore \quad f(z) = z^3 + 2z(1+i) + (2+3i)$$

$$\therefore \quad \boxed{f(z) = (z^3+2z+2)+i(2z+3)}$$

CONFORMAL MAPPING

Intro.: Suppose two curves c_1 , c_2 in the z-plane intersect at z_0 and the corresponding curves Υ_1 , Υ_2 in the w-plane intersect at w_0 by the transformation w= f(z).



If the angle between the two curves in the z-plane is same as the angle between the curves in the w-planes both in magnitude and in direction, then the transformation w = f(z) is said to be conformal mapping.

Definition:-

A transformation that preserves angles between every pair of curves through a point both in magnitude and sense of rotation is said to be conformal at that point.

CONFORMAL MAPPING

Isogonal Transformation:-

The transformation which preserves angle between every pair of curves in magnitude and not in direction(sense) is called an isogonal transformation.

Theorem:-

If $f(z)$ is analytic and $f'(z) \neq 0$ in a region R of the z-plane then the mapping performed by $w=f(z)$ is conformal at all points of R.

CONFORMAL MAPPING

Critical points:-

The point at which the mapping w=f(z) is not conformal, i.e., f ‘(z) = 0 is called a critical point of the mapping.

$$\underline{Eg.}: \quad Consider \quad w = f(z) = \sin z$$

$$\therefore \quad f'(z) = \cos z$$

$$\Rightarrow \quad f'(0) = 0\,,\ when\ z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, ...$$

$$i.e., \quad z = \frac{(2n-1)\pi}{2},\ where\ n\ is\ an\ integer,$$

which are the critical points of the given transformation.

Standard Transformations

- **Translation**

  Maps of the form $z \to z + k$, where $k \in C$

- **Magnification and rotation**

  Maps of the form $z \to k z$, where $k \in C$

- **Inversion**

  Maps of the form $z \to 1 / z$

EXAMPLE FOR TRANSLATION

1) *Find the region of the w-plane into which the rectangular region in the z-plane bounded by the lines x=0, y=0, x=1, y=2 is mapped under the transformation w=z+2-i .*

Solution:- *Given :* $w = z + 2 - i$

$\rightarrow \quad (u+iv) = (x+iy) + (2-i)$

$= (x+2) + i\ (y-1)$

Equating real and imaginary parts, weget,

$u = x + 2$ and $v = y - 1$

EXAMPLE FOR TRANSLATION

Given boundary lines are:

$x = 0$

$y = 0$

$x = 1$

$y = 2$

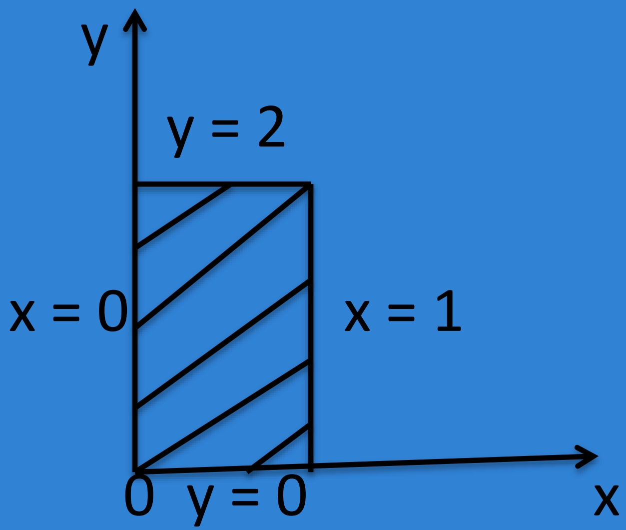
Transformal boundary lines are:

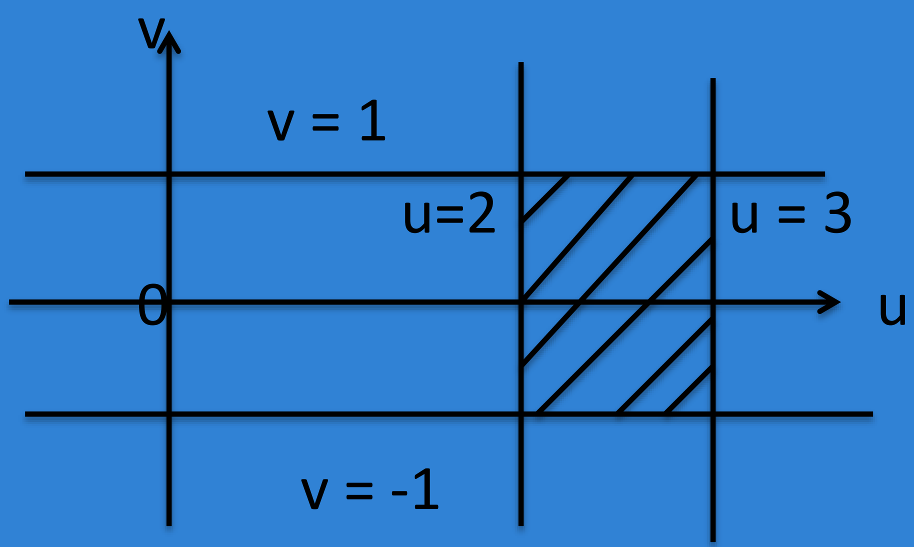
$u = 2$

$v = -1$

$u = 3$

$v = 1$





MAGNIFICATION AND ROTATION

Let $w = a z$, where $a \neq 0$

If $a = |a|\, e^{(i\,\alpha)}$ and, $z = |z|\, e^{(i\,\theta)}$, then

$w = |a|\,|z|\, e^{(i\,\theta + \alpha)}$

The image of z is obtained by rotating the vector z through the angle α and magnifying or contracting the length of z by the factor $|a|$.

Thus the transformation $w = a z$ is referred to as a **rotation** or **magnification**.

EXAMPLE FOR MAGNIFICATION

2) *Determine the region R of the w plane into which the triangular region D enclosed by the lines $x = 0$, $y = 0$, $x + y = 3$ is transformed under the transformation $w = 2z$.*

Solution:

Let $w = u + i v$; $z = x + i y$

Given: $w = 2z$

i.e., $u + i v = 2(x + i y)$

i.e., $u = 2x$; $v = 2y$ and $u + v = 2(x + y)$

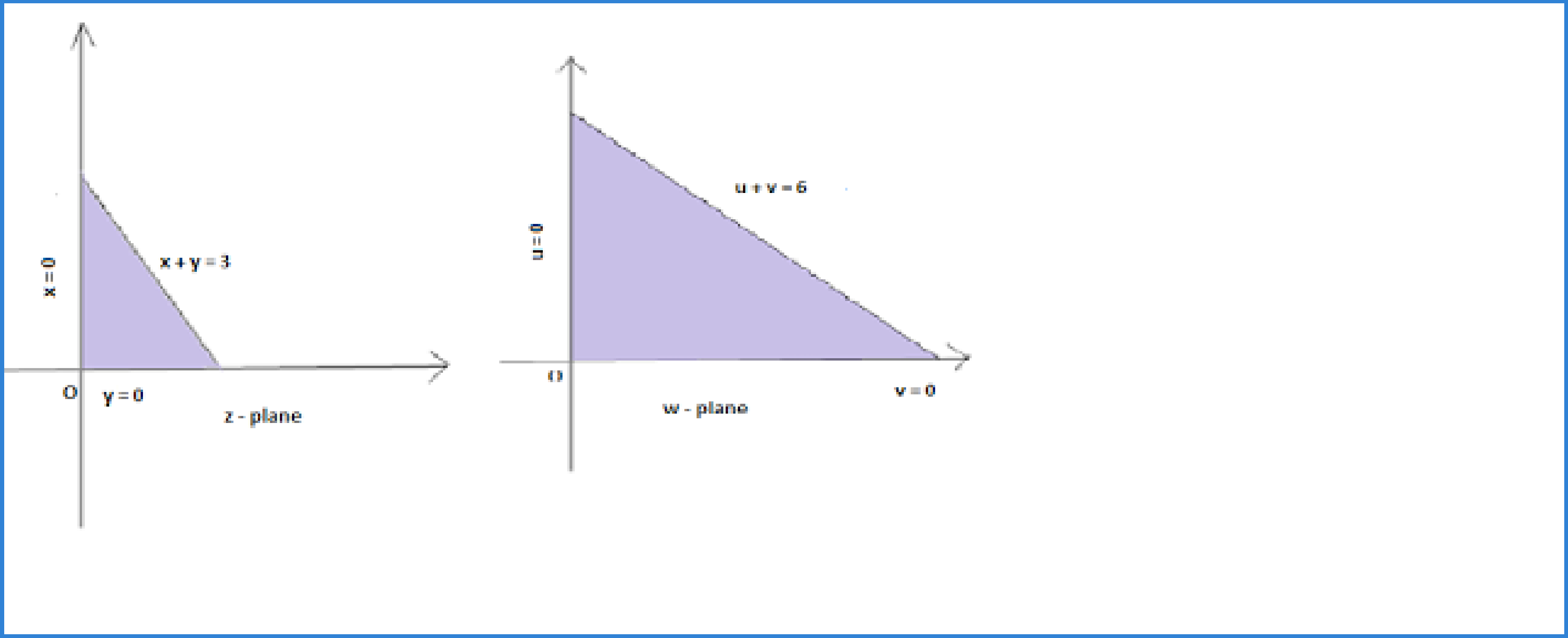
EXAMPLE FOR MAGNIFICATION

When $x = 0$, $u = 0$

$y = 0$, $v = 0$

$x + y = 3$, $u + v = 6$

Thus the transformation $w = 2z$ maps a triangle in the z-plane into a 2-times magnified triangle in the w-plane.



EXAMPLE FOR ROTATION

3) Consider the transformation w = $e^{i\pi/4}$ z and determine the region in the w-plane corresponding to triangle region bounded by the lines x=0 , y=0 , x+y=1.

$$\underline{Solution} : -$$

$$\underline{Given} : w = e^{i\pi/4}\, z$$

$$\therefore \quad (u+iv) = e^{i\pi/4}\,(x+iy)$$

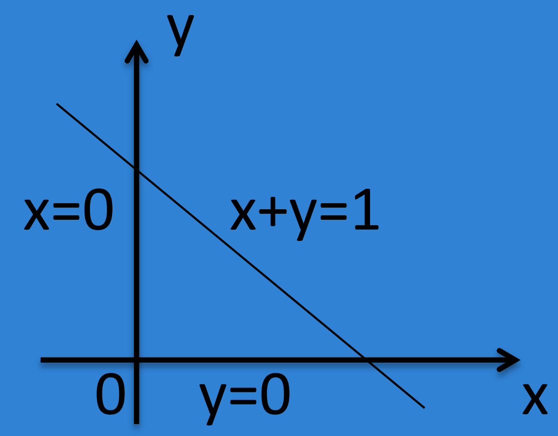
$$= \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)(x+iy)$$

$$= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy)$$

$$= \left(\frac{x-y}{\sqrt{2}}\right) + i\left(\frac{x+y}{\sqrt{2}}\right)$$

EXAMPLE FOR ROTATION

$$\therefore \quad \boxed{u = \frac{x-y}{\sqrt{2}}} \quad and \quad \boxed{v = \frac{x+y}{\sqrt{2}}}$$

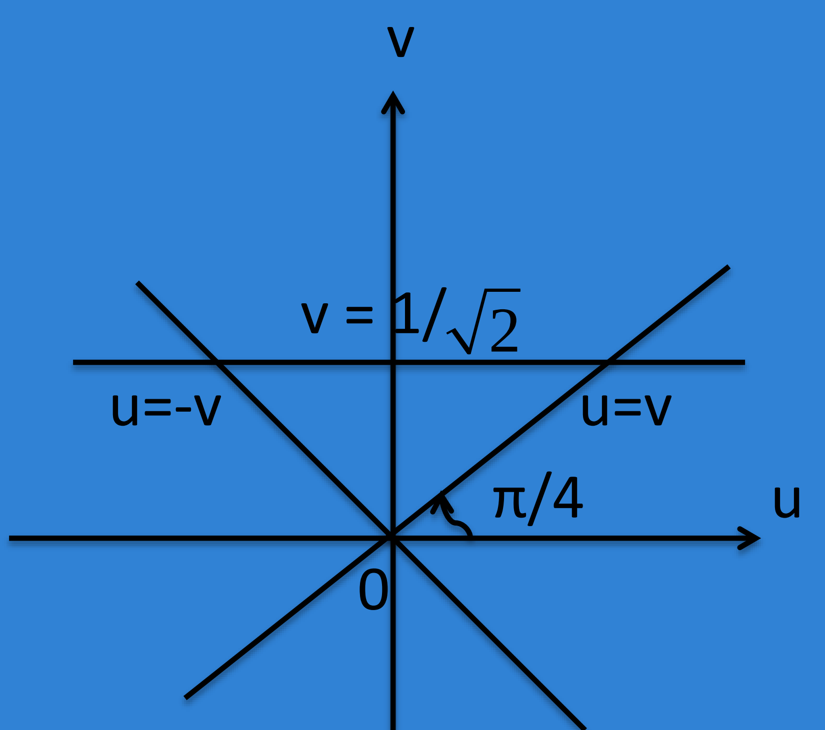


$\underline{\underline{when\ x=0,}}$ $\quad u = \frac{-y}{\sqrt{2}} \quad and \quad v = \frac{y}{\sqrt{2}}$

$$\Rightarrow y = -\sqrt{2}\, u \ and \ y = \sqrt{2}\, v$$

$$\Rightarrow -\sqrt{2}\, u = \sqrt{2}\, v$$

$$\Rightarrow \boxed{u = -v}$$



$\underline{\underline{when\ y = 0,}}$ $\quad u = \frac{x}{\sqrt{2}} \quad and \quad v = \frac{x}{\sqrt{2}}$

$$\Rightarrow \boxed{u = v}$$

$\underline{\underline{when \quad x + y = 1}}$ $\quad \Rightarrow \boxed{v = \frac{1}{\sqrt{2}}}$

EXAMPLE FOR ROTATION

The region in the z-plane is mapped on to the region bounded by u = -v, u = v, $v = \frac{1}{\sqrt{2}}$ in the w-plane.

$$\therefore \quad \textit{The mapping } w = ze^{i\pi/4} \textit{ performs a rotation}$$
$$\textit{of } R \textit{ through an angle } \pi/4.$$

INVERSE TRANSFORMATION

The Reciprocal Transformation w = 1/z

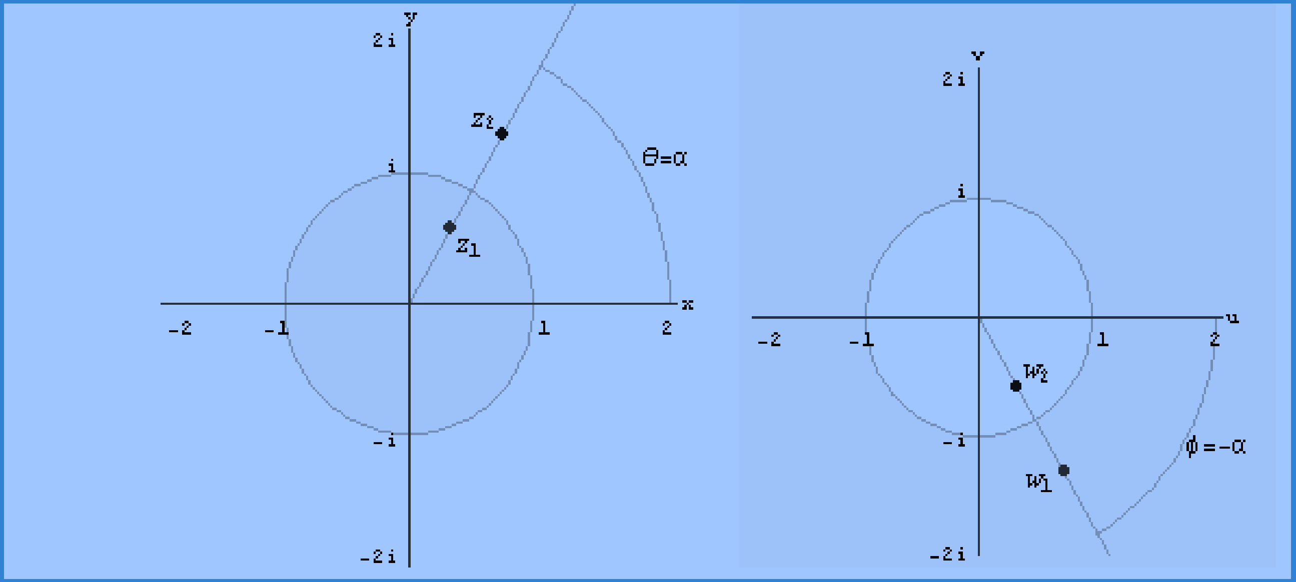
The mapping $w = \frac{1}{z}$ is called the reciprocal transformation and maps the z-plane one-to-one and onto the w-plane except for the point z=0, which has no image, and the point w=0, which has no preimage or inverse image. Use the exponential notation $w = \rho e^{i\phi}$ in the w-plane. If , $z = re^{i\theta} \neq 0$ we have $w = \rho e^{i\phi} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$.

.

INVERSE TRANSFORMATION

The geometric description of the reciprocal transformation is now evident. It is an inversion (that is, the modulus of $\frac{1}{z}$ is the reciprocal of the modulus of z) followed by a reflection through the x axis. The ray , $r > 0, \theta = \alpha$ is mapped one-to-one and onto the ray $\rho > 0, \phi = -\alpha$. Points that lie inside the unit circle $C_1(0) = \{z : |z| < 1\}$ are mapped onto points that lie outside the unit circle and vice versa. The situation is illustrated in Figure.

INVERSE TRANSFORMATION



EXAMPLE OF INVERSE TRANSFORMATION

1) Show that the image of the right half plane $A = \left\{z : \text{Re}(z) \geq \frac{1}{2}\right\}$ under the mapping $w = f(z) = \frac{1}{z}$ is the closed disk $\overline{D_1(1)} = \{w : |w - 1| \leq 1\}$ in the w-plane.

Solution:-

$$u + iv = w = f(z) = \frac{1}{z}$$

Then,

$$z = f^{-1}(w) = \frac{1}{w}$$

EXAMPLE OF INVERSE TRANSFORMATION

$$u + i v = w = f(z) \in \overline{D_1(1)}$$

$$\Longleftrightarrow f^{-1}(w) = x + i y \in A$$

$$\Longleftrightarrow \frac{1}{u + i v} = x + i y \in A$$

$$\Longleftrightarrow \frac{u}{u^2 + v^2} + i \frac{-v}{u^2 + v^2} = x + i y \in A$$

$$\Longleftrightarrow \frac{u}{u^2 + v^2} = x \geq \frac{1}{2}, \quad \text{and} \quad \frac{-v}{u^2 + v^2} = y$$

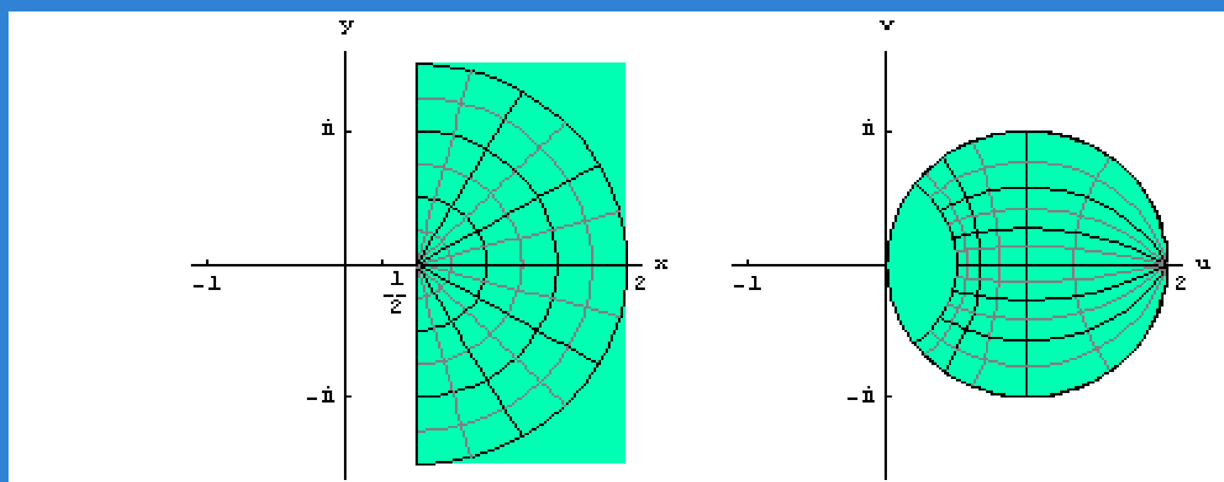
$$\Longleftrightarrow \frac{u}{u^2 + v^2} \geq \frac{1}{2}$$

$$\Longleftrightarrow u^2 - 2u + 1 + v^2 \leq 1$$

$$\Longleftrightarrow (u - 1)^2 + v^2 \leq 1$$

EXAMPLE OF INVERSE TRANSFORMATION

which describes the disk. As the reciprocal transformation is one-to-one, preimages of the points in the disk $\overline{D_1(1)}$ will lie in the right half-plane . Figure illustrates this result.



EXAMPLE OF INVERSE TRANSFORMATION

2) Find the images of the finite strips,

$$\frac{1}{4} \leq y \leq \frac{1}{2} \quad under\ the\ transformation\ w = \frac{1}{z}.$$

$\underline{Solution:-}$ $\quad Given: w = \frac{1}{z}$

$$\Rightarrow \quad z = \frac{1}{w}$$

$$i.e., \ x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore \quad x = \frac{u}{u^2+v^2} \ and \ y = \frac{-v}{u^2+v^2}$$

$$\downarrow\rightarrow (1) \qquad \downarrow\rightarrow (2)$$

EXAMPLE OF INVERSE TRANSFORMATION

$$\underline{\underline{Given}} : \underline{\underline{\frac{1}{4} < y < \frac{1}{2}}}$$

$$when \;\; y = \frac{1}{4} \;\; eqnuation \;\; (2) \;\; becomes,$$

$$\frac{1}{4} = \frac{-v}{u^2 + v^2}$$

$$\Rightarrow \;\; u^2 + v^2 = -4v$$

$$\Rightarrow \;\; u^2 + v^2 + 4v + 4 - 4 = 0$$

$$\Rightarrow \;\; \boxed{u^2 + (v+2)^2 = 4}$$

which is a circle whose centre at $(0, -2)$ *and radius is* 2 *in* $w-$ *plane.*

EXAMPLE OF INVERSE TRANSFORMATION

when $y = \frac{1}{2}$, *equation* (2) *becomes*,

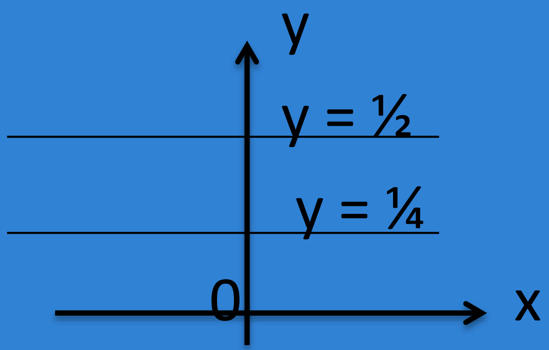
$$\frac{1}{2} = \frac{-v}{u^2 + v^2}$$

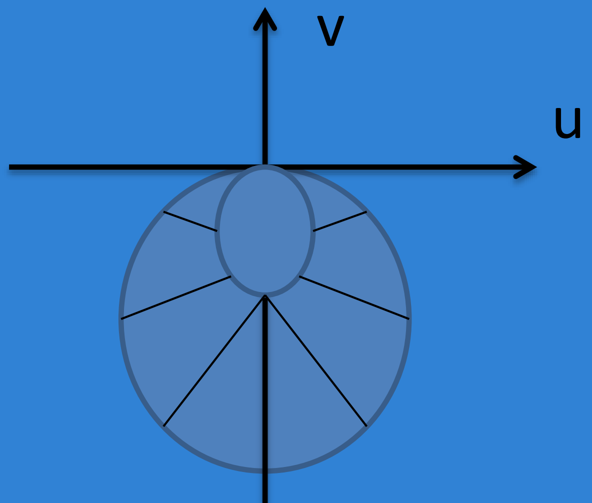
$$\Rightarrow \; u^2 + v^2 = -2v$$

$$\Rightarrow \; u^2 + v^2 + 2v + 1 - 1 = 0$$

$$\Rightarrow \; \boxed{u^2 + (v+1)^2 = 1}$$

which is a circle whose centre is at $(0,-1)$ *and radius is* 1.





BILINEAR TRANSFORMATION

Def.:–

The transformation $w = \dfrac{az+b}{cz+d}$, where a, b, c, d are complex constants and $ad - bc \neq 0$ is known as bilinear transformation.

Note:–

(i) A bilinear transformation is also called as Mobius transformation or a linear fractional transformation.

(ii) The inverse mapping of $w = \dfrac{az+b}{cz+d}$ is $z = \dfrac{-wd+b}{cw-a}$ is also called as a bilinear transformation.

BILINEAR TRANSFORMATION

Fixed points (or) Invariant points :-

If the image of a point z under a transformation w=f(z) is itself, then the point is called a fixed point or an Invariant point of the transformation.

Thus fixed point of the transformation w=f(z) is given by z = f(z).

$\underline{\underline{Eg.}}:\quad Let\ \ w = \dfrac{z}{z-2}\ ,\ find\ the\ fixed\ \ point\ (or)$

$invarient\ \ point.$

$\underline{Solution:-}\qquad put\ \ w = z$

$$then\ \ z = \frac{z}{z-2} \Rightarrow\ z^2 - 2z = z$$

$$\Rightarrow\ \ z(z-3) = 0$$

$$\Rightarrow z = 0,\ z = 3\ are\ two\ \ fixed\ \ points.$$

BILINEAR TRANSFORMATION

Definition of cross ratio:-

If z_1, z_2, z_3, z_4 are four points in the z-plane then the ratio $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$ is called the cross ratio of these points.

Cross Ratio Property of a bilinear transformation:-

The cross ratio of four points is invariant under a bilinear transformation.

i.e., If w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 respectively under a bilinear transformation then

$$\left(\frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}\right) = \left(\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}\right)$$

BILINEAR TRANSFORMATION

Note:-

The bilinear transformation which transforms the points z_1 , z_2 , z_3 of z-plane respectively into the points w_1,w_2,w_3 of w-plane is given by

$$\left(\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}\right) = \left(\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}\right)$$

BILINEAR TRANSFORMATION EXAMPLES

1) Find the bilinear transformation which maps the points $z = 0, -i, -1$ into $w = i, 1, 0$.

Solution:-

$$\underline{Given}: z_1 = 0, z_2 = -i, z_1 = -1$$

$$and \quad w_1 = i, \; w_2 = 1, w_3 = 0.$$

The bilinear transformationion is got by using the relation

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-i + 1)}{(0 + i)(-1 - z)}$$

BILINEAR TRANSFORMATION EXAMPLES

$$\Rightarrow (-i)(w-i)(1+z) = z(1-i)(-w)(i-1)$$

$$\Rightarrow -i - iwz - 1 - z = -2iwz$$

$$\Rightarrow -iw + iwz = 1 + z$$

$$\Rightarrow w(zi - i) = (1 + z)$$

$$\Rightarrow \quad w = \frac{1+z}{zi-i}$$

$$\Rightarrow \quad \boxed{w = \frac{1+z}{(-i)(1-z)}}$$

BILINEAR TRANSFORMATION EXAMPLES

2) *Find the bilinear transformation which transforms the points* $z = \infty, i, 0$ *into the points* $w = 0, i, \infty$ *respectively.*

Solution :–

Given : $z_1 = \infty, z_2 = i, z_3 = 0$

and $w_1 = 0, \ w_2 = i, w_3 = \infty$.

The bilinear transformationion is got by using the relation

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

BILINEAR TRANSFORMATION EXAMPLES

$$\frac{(w-w_1)(w_3)\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)(w_3)\left(1-\frac{w}{w_3}\right)} = \frac{(z_1)\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{(z_1)\left(1-\frac{z_2}{z_1}\right)(z_3-z)}$$

$$\Rightarrow \frac{(w-0)(0-1)}{(0-1)(i-0)} = \frac{(0-1)(i-0)}{(z-0)(0-1)}$$

$$\Rightarrow \frac{(-w)}{(-i)} = \frac{(-i)}{(-z)}$$

$$\Rightarrow \boxed{w = \frac{-1}{z}}$$

BILINEAR TRANSFORMATION

TRY IT

3) Find the bilinear transformation which maps the points,

   i) 1, -i , 2 onto 0, 2, i respectively.

   ii) -i,0,i into -1, i, 1 respectively.

   iii) 0, 1, ∞ into i, -1, -i respectively.