NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Contents:

CLASSIFICATION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Let a second order partial differential equation in the function u of the two independent variables x,y of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C(x,y)\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0..............(1)$$

The equation (1) is classified as elliptic , parabolic or hyperbolic at the points of a given region R depending on whether

$$B^2-4AC \text{ is less than Zero.......(elliptic equation)}$$
$$B^2-4AC \text{ is equal to Zero.......(Parabolic equation)}$$
$$B^2-4AC \text{ is greater than Zero.......(hyperbolic equation)}$$

Examples:

Elliptic type:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Parabolictype:

Onedimensi onalheatflowequation:

$$\frac{\partial^2 u}{\partial x^2}=a\frac{\partial u}{\partial t}$$

Hyperbolictype:

Onedimensi onalwaveequation

$$\frac{\partial^2 u}{\partial x^2}=a\frac{\partial^2 u}{\partial t^2}$$

Poissonseq uation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$$

Classify the following equations:

$$x^2 f_{xx} + (1-y^2) f_{yy} = 0 \text{ for } -1<y<1, -1<x<1$$

$A = x^2 \; B = 0 \; C = 1-y^2$

$B^2 - 4AC = -4x^2(1-y^2) = 4x^2(y^2-1)$. x^2 is always +ve in $-\alpha<x<\alpha$

In $-1<y<1$, y^2-1 is negative $\therefore B^2-4AC = -$ve $\therefore$ It is elliptic

If $x=0$, $B^2-4AC=0$, the equation is parabolic, when $y>1$

$y<-1$ then $B^2-4AC>0$ then the equation is hyperbolic.

Exercise:

1. $x f_{xx} + y f_{yy} = 0 \quad x>0, y>0$

$2 f_{xx} - 2 f_{xy} = 0 \quad x>0, y>0$

Difference Quotient

A difference quotient is the quotient obtained by dividing the differences between two values of a function by the difference between the two corresponding values of the independent variable.Thus for a function f(x) of a single variable x, the difference quotient is $\frac{f(x+h)-f(x)}{h}$ whose limiting value is the derivative of f(x) with respect to x. $l_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$ (i.e)

Thus a difference quotient is an approximation to the derivative.

Partial difference quotients of the second order is constructed with reference to a networkof points in the XY-plane be divided into a network or lattice of rectangles, by drawing two families of parallel lines at x=ih, y=jk,I,j = 0,1,2,….

			(x,y+2k)	
			(x,y+k)	
	(x+h,y)	(x-h,y)	(x,y)	(x+h,y)
			(x,y-k)	
			(x,y-2k)	

Fig (1)

The points of intersection of these families of lines are called lattice points.

From the fig, the forward first diff quotient of u(x,y) w.r.t x is

$$u_x = \frac{u(x+h,y)-u(x,y)}{h} \ldots\ldots\ldots\ldots\ldots(1)(\mathit{forwarddifference})$$

and the backward first difference quotient of u(x,y) w.r.t x is

$$u_x = \frac{u(x,y)-u(x-h,y)}{h} \ldots\ldots\ldots\ldots\ldots(2)(\mathit{Backwarddifference})$$

We can also have

$$u_x = \frac{u(x+h,y)-u(x-h,y)}{2h} \ldots\ldots\ldots\ldots\ldots(2a)$$

The second difference quotient of u(x,y) w.r.t x is the difference quotient of the first difference quotients (1) and (2)

Hence we have

$$u_{xx}=\frac{u(x+h,y)-2u(x,y)+u(x-h,y)}{h^2}\text{............}(3)$$

Similarly the first and the second difference quotients of u(x,y) w.r.t y are found exactly in the same manner and

$$u_y=\frac{u(x,y+k)-u(x,y)}{k}\text{............}(\text{ forwarddifference})\text{............}(4)$$

$$u_y=\frac{u(x,y)-u(x,y-k)}{k}\text{............}(\text{Backwarddifference})\text{............}(5)$$

$$u_y=\frac{u(x,y+k)-u(x,y-k)}{2k}\text{..................}(5a)$$

$$u_{yy} = \frac{u(x, \ y+k) - 2u(x, \ y) + u(x, \ y-k)}{k^2} \quad \text{...............(6)}$$

The above difference quotients can also be written in anotherway by drawing two sets of parallel lines x=ih and y=jk, i,j= 0,1,2,…. Aa shown by taking h=k=1.

The point (i,j) is called the grid point and is surrounded by the neighboring grid pts as shown in the fig(2)

X=i

		(i-1,j+1)	(I,j+1)	(i+1,j+1)
		(i-1,j)	(I,j)	(i+1,j)
K=1		(i-1,j-1)	(I,j-1)	(i+1,j-1)

Y=j

h=1

Fig (2)

We have

$$u(x,y)=u(ih,jk)=u_{i,j}$$
$$u(x+h,y)=u(ih+h,jk)=u_{i+1,j}$$
$$u(x-h,y)=u(ih-h,jk)=u_{i-1,k}$$
$$u(x,y+k)=u(ih,jk+k)=u_{i,j+1}$$
$$u(x,y-k)=u(ih,jk-k)=u_{i,j-1}.....andsoon$$

Using these notations the equations (1),(2),(2a),(3),(4) (5),(6) can be written as

$$u_x = \frac{u_{i+1,j} - u_{i,j}}{h} (\, forward difference) \ldots\ldots\ldots\ldots (7)$$

$$u_x = \frac{u_{i,j} - u_{i-1,j}}{h} (Backward difference) \ldots\ldots\ldots\ldots (8)$$

$$u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \ldots\ldots\ldots\ldots (8a)$$

$$u_y = \frac{u_{i,j+1} - u_{i,j}}{k} (\, forward difference) \ldots\ldots\ldots\ldots (9)$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k} (backward difference) \ldots\ldots\ldots\ldots (10)$$

$$u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \ldots\ldots\ldots\ldots (10a)$$

$$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} \ldots\ldots\ldots\ldots (11)$$

$$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i+1,j}}{k^2}$$

Solution of elliptic equations

The laplace equation in two dimensions $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

This equation can also be written as

$$u_{xx}+u_{yy}=0$$

Let us solve this diff eqn at the interior points of the given region R with the given values of u at the boundary R, For we convert eqn (1) into a diff eqn

We know that

$$u_{xx}=\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}\text{..........}(2)$$

$$u_{yy}=\frac{u_{i,j-1}-2u_{i,j}+u_{i+1,j}}{k^2}\text{.........}(3)$$

$$sub(2)and(3)in(1)$$

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i+1,j}}{k^2}=0 \ldots\ldots.(4)$$

If we consider a square mesh with h=k=1, then eqn (4) becomes

$$u_{i-1,j}-4u_{i,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}=0 or$$

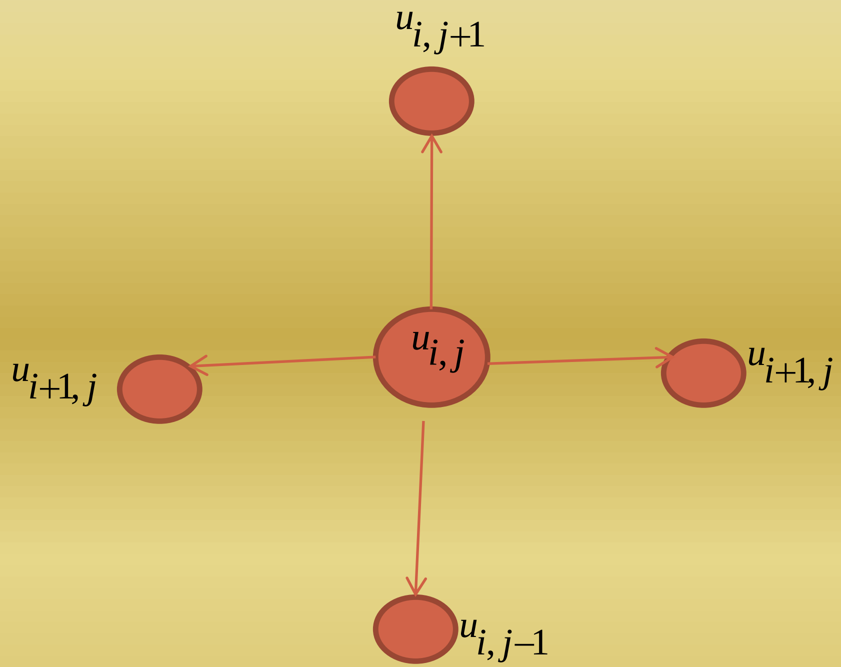
$$u_{i,j}=\frac{1}{4}\left(u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}\right)\ldots\ldots.(5)$$

This formula is called the standard five point formula. This formula can be exhibted in the figure(1)

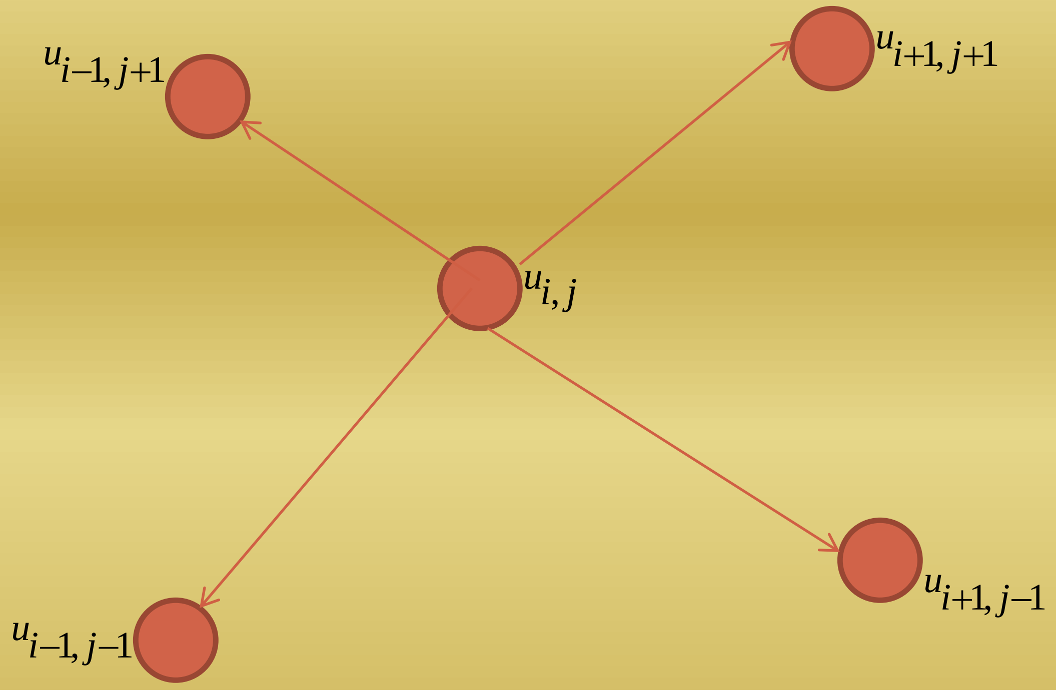
We can also use the formula

$$u_{i,j}=\frac{1}{4}\left(u_{i-1,j}-1+u_{i+1,j-1}+u_{i+1,j+1}+u_{i-1,j-1}\right)$$

Which is called diagonal five point formula exhibted in the figure (2)



Standard five point formula



Diagonal five point formula

Solution of Laplace equation

Let us consider the process of solving the laplace equation in two variables namely: $\frac{\partial 2u}{\partial x2}+\frac{\partial 2u}{\partial y2}=0$ with the given boundary conditions. For simplicity we assume that the function u(x,y) is required over a rectangular area. We therefore divide the area with a network of squares of side h

The boundary values of the desired function are given and we denote them by $a_1, a_2, a_3, \ldots\ldots, a_n,$ as indicated in the figure. The values of the required function at the interior lattice points namely $a_1, a_2, a_3, \ldots\ldots, a_n,$ are unknown.

$$u_5=\frac{1}{4}\left(a_{15}+a_3+a_7+a_{11}\right)$$

After finding u_5 we can find the initial values at the centres of the four larger inner squares by taking the means of the values at the ends of the diagonals passing through them.

a_1
a_2
a_3
a_4
a_5
a_6
a_7
a_8
a_9
a_{10}
a_{11}
a_{12}
a_{13}
a_{14}
a_{15}
a_{16}
u_1
u_2
u_3
u_4
u_5
u_6
u_7
u_8
u_9

$$u_1=\frac{1}{4}\left(a_1+a_3+u_5+a_{15}\right)$$
$$u_3=\frac{1}{4}\left(u_5+a_3+a_5+a_7\right)$$
$$u_7=\frac{1}{4}\left(a_{15}+u_5+a_{11}+a_{13}\right)$$
$$u_9=\frac{1}{4}\left(u_5+a_7+a_9+a_{13}\right)$$

After finding u_2, u_4, u_6, u_8 we can find the initial values for u_2, u_4, u_6, u_8 as given below

$$u_2=\frac{1}{4}\left(u_1+a_3+u_3+u_5\right)$$
$$u_4=\frac{1}{4}\left(a_{15}+u_1+u_5+u_7\right)$$
$$u_6=\frac{1}{4}\left(u_5+u_3+a_7+u_9\right)$$
$$u_8=\frac{1}{4}\left(u_7+u_5+u_9+a_{11}\right)$$

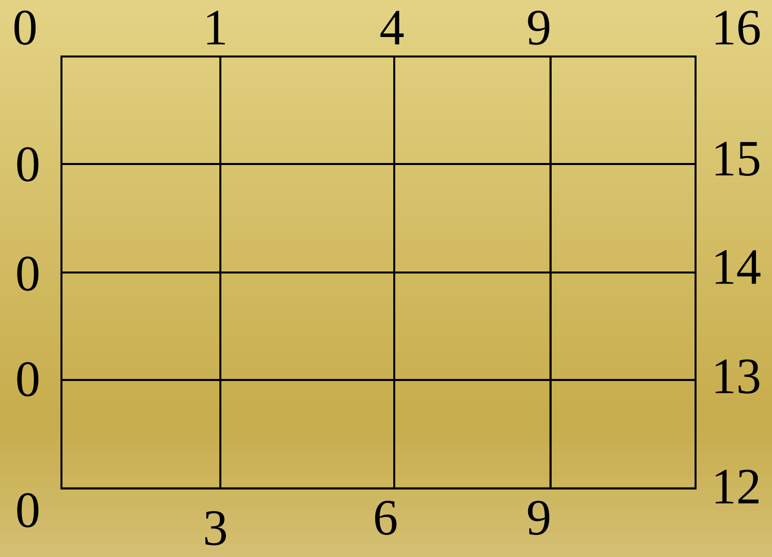
After finding all u’s we can improve their values by using an iteration formula given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left(u_{i-1,j}^{(n+1)} + u_{i,j+1}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)}\right)$$

Which is called the liebmann’s iteration process

Example

Find by Liebmann's Method the values at the interior lattice points of a square plate of the harmonic function *u* whose boundary values are given in the figure.



Solution

Using cross averaging and diagonal averaging, the successive iterations yield the following values:

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
2.5000	5.625	10.000	3.1250	6.0000	9.8750	3.0000	6.1250	9.5000
2.4375	5.6094	9.8711	2.8594	6.1172	9.8721	2.9948	6.1530	9.5063
2.3672	5.5888	9.8652	2.8698	6.1209	9.8731	3.0057	6.1582	9.5078
2.3647	5.5877	9.8652	2.8728	6.1230	9.8740	3.0078	6.1597	9.5084
2.3651	5.5883	9.8656	2.8740	6.1240	9.8745	3.0084	6.1602	9.5087

Poisson's equation

The partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=F(x,y) \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Is called the poisson's equation where F(x,y) is a given function of x and y.

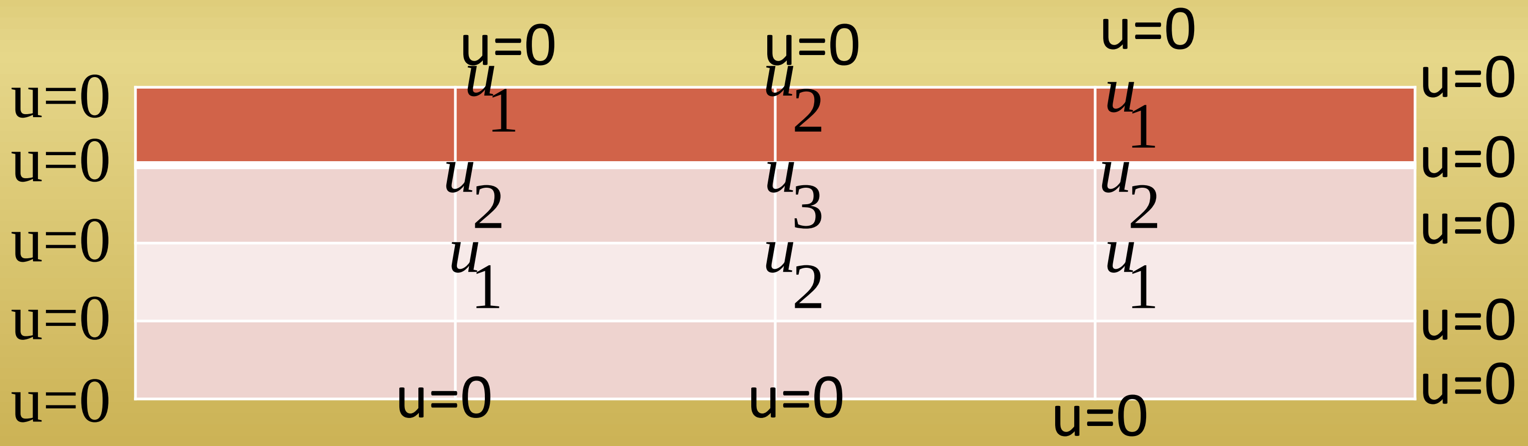
Replacing the derivatives in eq (1) by the difference expressions at the points x=ih and y=jk (here j=k)we get

$$\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2}=F(ih,jh)$$

$$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}=h^2F(ih,jh).....(2)$$

By applying (2) at each mesh point, we will get a system of linear equations in the pivotal values i,j

Solve the Poisson's equation $\nabla^2 u = 8x^2y^2$ for the square mesh of the given figure with u(x,y)=0 on the boundary and mesh length=1



Solution:

Given h=1, From the standard five point formula we get

$$u_{i-1,j}+u_{i+1,j}+u_{i,j-1}+u_{i,j+1}-4u_{i,j}=8i^2j^2 \ldots\ldots\ldots\ldots\ldots(1)$$

u_1

Now for the point (where i=1, j=-1), the equation (1) gives

$$0+u_2+0+u_2-4u_1=8$$

$$u_2-2u_1=4 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

for the point u_2 (where i=1, j=0), the equation (1) gives

$$u_1+u_1+0+u_3-4u_2=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$2u_1+u_3-4u_2=0$$

u_3

for the point (where i=0, j=0), the equation (1) gives

$$u_2+u_2+u_2+u_2-4u_3=0$$

$$u_2-u_3=0 \ldots\ldots\ldots(4)$$

The above equations (2),(3),(4) can be written as

$$2u_1 - u_2 = -4 \ldots\ldots\ldots\ldots(5)$$

$$2u_1 - 4u_2 + u_3 = 0 \ldots\ldots\ldots\ldots(6)$$

$$u_2 - u_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

From (7) we get $u_2 = u_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$

Substituting (8) in (6) we get

$$2u_1 - 3u_2 = 0 \ldots\ldots\ldots\ldots(9)$$

From (5) $2u_1 - u_2 = -4 \ldots\ldots\ldots\ldots(10)$

Solving (9) and (10)

$$u_2 = -2$$

$$u_1 = -3$$

$$u_3 = -2$$

$$\therefore u_1 = -3$$

$$u_2 = -2$$

$$u_3 = -2$$

Bender – Schmidt Method

Consider the equation $u_{xx} = au_t$ with boundary conditions

$u(0, t) = T_0$ and $u(1, t) = T_1$

The initial condition is u (x, 0) = f(x). Let h be the spacing

for x and k be the spacing for t.

Using finite difference approximation for derivatives,

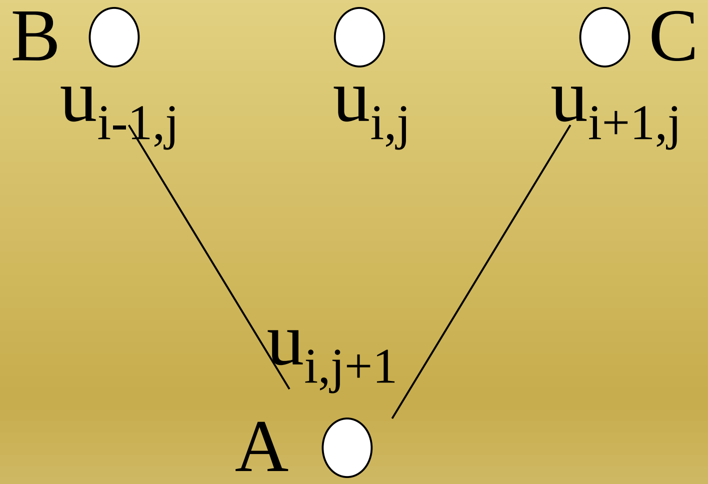
and applying boundary conditions and considering the

special case $\lambda = \dfrac{k}{h^2 a} = \frac{1}{2}$, we get

Bender – Schmidt Method

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

This formula means that the value of u at $x = x_i$ and $t = t_{j+1}$ is the arithmetic mean of the values of u at the surrounding points x_{i-1} and x_{i+1} at the previous time t_j.



Value at A = ½ (Value at B + Value at C)

Given $\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$ f(0,t)=f(5,t)=0,f(x,0)= $x^2(25-x^2)$
find f in the range taking h=1 and upto 5 seconds

Solution: $k = \frac{a}{2}h^2$

To use schimdth method ,

Here a=1, h=1 hence k=1/2

Step size of time =1/2

Step size of x= 1

F(0,0)=0,f(1,0)=24, f(2,0)=84, f(3,0)=144,f(4,0)=144, f(5,0)=0

We have

$$u_{i,j+1} = \frac{1}{2}\left(u_{i-1,j} + u_{i+1,j}\right)$$

→ x direction

↓ t-direction

j \ i	0	1	2	3	4	5
0	0	24	84	144	144	0
½	0	42	84	114	72	0
1	0	42	78	78	57	0
1·5	0	39	60	67·5	39	0
2	0	30	53·25	49·5	33·75	0
2·5	0	26·625	39·75	43·5	24·75	0
3	0	19·875	35·0625	32·25	21·75	0
3·5	0	17·5312	26·0625	28·4062	16·125	0
4	0	13·0312	22·9687	21·0938	14·2031	0
4·5	0	11·4843	17·0625	18·5859	10·5469	0
5	0	8·5312	15·0351	13·8047	9·2929	0

Example

Find the value of the function u (x,t) satisfying the equation $$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

The boundary conditions are $u\,(0,t) = 0 = u\,(8,\,t)$

and the initial condition is $u(x,0) = 4x - \frac{1}{2}x^2$

at the points x = i, i = 0, 1, 2, 3, 4 and t = (1/8) j, j = 0, 1, 2, 3, 4, 5.

Solution

Given a = $\frac{1}{4}$, h = 1, λ = $\frac{k}{h2a}$ = 4k. If λ = $\frac{1}{2}$ then k = $\frac{1}{8}$.

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5525	6	5.5625	4.25	2.125	0
5	0	2.125	3.9875	5.125	5.5625	5.125	3.9875	2.125	0

Crank- Nicholson difference scheme for solving Parabolic equations

Let $u_{xx} = au_t$.....(1) be the given parabolic equation to be solved with the boundary conditions $u(0,t) = T_0$ and $u(1,t) = T_l$ and the initial condition u(x,0)=f(x)

We know that the point $u_{i,j}$ the finite difference approximation for u_{xx} is

$$u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \text{.......(2)}$$

At the point $u_{i,j+1}$ the finite difference approximation for u_{xx} is

$$u_{xx} = u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} \text{.......(3)}$$

Taking the average of (2) and (3)

$$u_x = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} \quad .....(4)$$

We know that the forward difference approximation for u_t is

$$u_t = \frac{u_{i,j+1} - u_{i,j}}{k}$$

Sub (4) and (5) in (1) we get

$$\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2}$$

$$= a\left(\frac{u_{i,j+1} - u_{i,j}}{k}\right)$$

$$\frac{k}{2h^2 a}\left[u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

$$= u_{i,j+1} - u_{i,j} \quad ....(6)$$

Putting in $\frac{k}{h^2 a} = \lambda$ in (6) we get

$$\frac{\lambda}{2}\left(u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}+u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)$$

$$=u_{i,j+1}-u_{i,j}$$

$$\frac{\lambda}{2}u_{i+1,j+1}-\lambda u_{i,j+1}+\frac{\lambda}{2}u_{i-1,j+1}-u_{i,j+1}$$

$$=-\frac{\lambda}{2}u_{i+1,j}+\lambda u_{i,j}-\frac{\lambda}{2}u_{i-1,j}-u_{i,j}+\frac{\lambda}{2}u_{i+1,j+1}-(\lambda+1)u_{i,j+1}+\frac{\lambda}{2}u_{i-1,j+1}$$

$$=-\frac{\lambda}{2}u_{i+1,j}+(\lambda-1)u_{i,j}-\frac{\lambda}{2}u_{i-1,j}$$

$$\lambda(u_{i+1,j+1}+u_{i-1,j+1})-(2\lambda+2)u_{i,j+1}=(2\lambda-2)u_{i,j}-\lambda(u_{i+1,j}+u_{i-1,j})$$

Which is the crank –nicholson formula

Example 2:

Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ subject to the conditions u(x,0)=0, u(0,t)=0 and u(1,t)=t. Compute u for t= 1/8 in two steps using crank- nicholson formula

Solution:

The given equation can be written as $u_{xx}=u_t$ ………….(1)

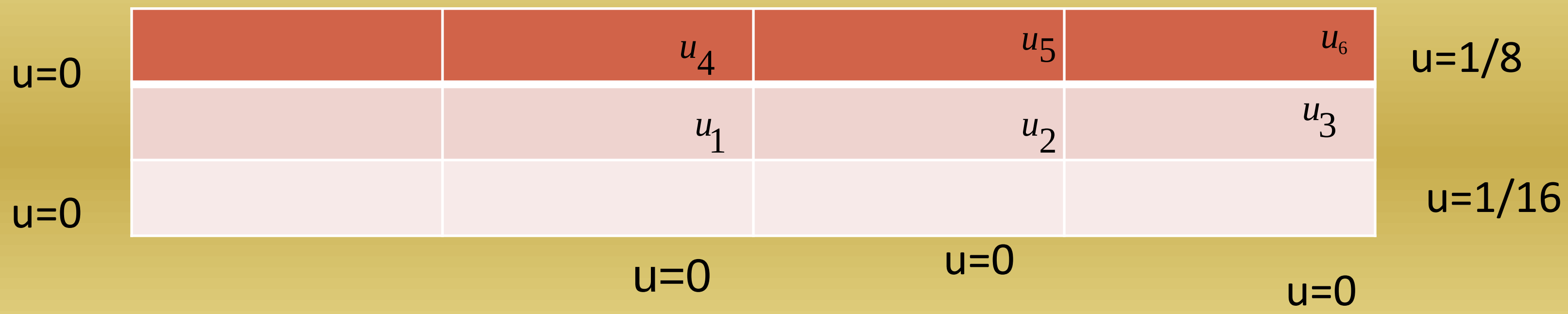
Here a= 1 Take h = ¼, k = 1/ 16

$$\lambda=\frac{k}{h^2 a}=\frac{\frac{1}{16}}{\frac{1}{16}}=2$$

$$\lambda=1$$

The crank nicholson scheme corresponding to $\lambda=1$ is

$$u_{i+1,j+1}-4u_{i,j+1}+u_{i-1,j+1}=-u_{i+1,j}-u_{i-1,j} \ldots\ldots\ldots\ldots\ldots(2)$$



Applying (2) at the mesh points u_1, u_2, u_3 we get

$$u_2 - 4u_1 + 0 = 0 \ldots\ldots\ldots\ldots (3)$$
$$u_3 - 4u_2 + u_1 = 0 \ldots\ldots\ldots\ldots (4)$$
$$\frac{1}{16} - 4u_3 + u_2 = 0 \ldots\ldots\ldots\ldots (5)$$

Solving (3), (4) and (5) we get

$u_1 = 0.001116$

$u_2 = 0.004464$

$u_3 = 0.01674$

Applying (2) at the mesh points u_4, u_5, u_6 we get

$$u_5 - 4u_4 + 0 = -0.004464 \ldots\ldots\ldots (6)$$

$$u_6 - 4u_5 + u_4 = 0.01674 - 0.001116 \ldots\ldots\ldots (7)$$

$$\frac{1}{8} - 4u_6 + u_5 = -\frac{1}{16} - 0.004464 \ldots\ldots\ldots (8)$$

Solving (6) ,(7),(8) we get

$u_4 = 0.005899$

$u_5 = 0.019132$

$u_6 = 0.052771$

Solution of hyperbolic equations

The well known hyperbolic equation is one dimensional wave equation and is given by $a^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0$ This equation can also be written as

$$a^2u_{xx}-u_{tt}=0..........(1)$$

Let us solve (1) with the boundary conditions

u(0,t) = 0……….(2)

u(l,t) = 0 ………(3)

And the initial conditions u(x,0) =f(x) …..(4)

$$u_t(x,0).......(5)$$

To solve (1) we have to replace the differential coefficient in (1) by its difference quotients

We know that

$$u_{xx}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}............(6)$$

$$u_{tt}=\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}............(7)$$

Substituting (6),(7) in (1) we get

$$a^2\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right)-\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{k^2}\right)=0$$

$$u_{i+1,j}-2u_{i,j}+u_{i-1,j}=\frac{h^2}{k^2a^2}\left(u_{i,j+1}-2u_{i,j}+u_{i,j-1}\right)=0\text{...........}(8)$$

Putting k/h= λ in (8) we get

$$\lambda^2a^2\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)=u_{i,j+1}-2u_{i,j}+u_{i,j-1}$$

$$u_{i,j+1}=\lambda^2a^2\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)+2u_{i,j}-u_{i,j-1}$$

$$u_{i,j+1}=2\left(1-\lambda^2a^2\right)u_{i,j}+\lambda^2a^2\left(u_{i+1,j}+u_{i-1,j}\right)-u_{i,j-1}\text{.............}(9)$$

This scheme is called an explicit scheme for the solution of the wave equation

Solve numerically $4u_{xx}=u_{tt}$, with the boundary conditions u(0,t)= 0, u(4,t)= 0 and the initial conditions $u_t\left(x,0\right)$ and u(x,0)= x(4-x), taking h=1 for 4 time steps

Solution:

Since $a^2=4, h=1, k=\frac{h}{a}=\frac{1}{2}$ ....(1)

Taking k=1/2 we use the formula

$$u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1} \qquad \text{.............(2)}$$

From u(0,t)= 0 implies u along x=0 are all zero
From u(4,t)= 0 implies u along x = 4 are all zero

u(x,0)= x(4-x) implies that

u(0,0)=0, u(1,0)=3 , u(2,0)=4, u(3,0)=3

Now, we fill up the row t=0 using the above values

$$u_t(x,0)=0 \Rightarrow u_{i,1}=\frac{u_{i+1,0}+u_{i-1,0}}{2} \quad \text{............(3)}$$

Now we draw the table for that we require

$$u_{1,1} = \frac{u_{2,0} + u_{0,0}}{2} = \frac{4+0}{2} = 2$$
$$u_{2,1} = \frac{u_{3,0} + u_{1,0}}{2} = \frac{3+3}{2} = 3$$
$$u_{3,1} = \frac{u_{4,0} + u_{2,0}}{2} = 2$$
$$u_{4,1} = 0$$

t \ x	0	1	2	3	4
0	0	3	4	3	0
0.5	0	2	3	2	0
1	0	0	0	0	0
1.5	0	–2	–3	–2	0
2	0	–3	–4	–3	0
2.5	0	–2	–3	–2	0
3	0	0	0	0	0
3.5	0	2	3	2	0
4	0	3	4	3	0

Period is 4 seconds or 8(k) = 8(1/2)=4 sec

Example

Tabulate the pivotal values for the equation

$$16\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$$

Given that

$u(0, t) = 0$, $u(5, t) = 0$, $u(x, 0) = x^2(5 - x)$ and $u_t(x, 0) = 0$

Assume $h = 1$.

Solution

j \ i	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	-2	0
3	0	2	-2	-8	-7	0
4	0	-9	-14	-11	-6	0
5	0	-16	-18	-12	-4	0
6	0	-9	-14	-11	-6	0
7	0	2	-2	-8	-7	0
8	0	7	8	2	-2	0
9	0	6	11	14	9	0
10	0	4	12	18	16	0

$$u_{i,j+1} = u_{i-1,\, j} + u_{i+1,\, j} - u_{i,\, j-1}$$