PROBABILITY AND QUEUING THEORY

Markov Process

An Interesting model of a random process is the one in which the value of the random process depends only upon the most recent previous value and is independent of all values in the more distant past . Such a model is called a Markov model and is often described by saying that a Markov Process is one in which the future value is independent of the past value, given the present value.

Simple Example of A Markov Chain

Consider the experiment of tossing a fair coin a number of times . The possible outcomes at each trial are two-head with probability ½ and tail with probability ½ .If we denote the outcome of the n^{th} toss , which is a RV ,by X_n and the outcomes head and tail by 1and 0 respectively ,then

$$P[X_n = 1] = ½ \text{ and } P[X_n = 0] = ½ \ ; n= 1,2,......$$

Thus we have a sequence of independent RVs $X_1, X_2, ...$ Since the trails are independent and hence the outcome of the n^{th} trial does not depend in any way on the previous trails.

Consider now the RV that represents the total number

of heads in the first n trials and is given by

$$S_n = X_1 + X_2 + \ldots\ldots + X_n.$$

The possible values of S_n are 0,1,2,....n. If

$$S_n = k (k=0,1,2,\ldots.n),$$

then the RV $S_{n+1} (= S_n + X_{n+1})$ can assume only two possible values, namely k+1 [if the $(n+1)^{th}$ results in a head] and k [if the $(n+1)^{th}$ trial results in a tail].

Thus $P\{S_{n+1} = k+1 / S_n = k\} = ½$

$P\{S_{n+1} = k / S_n = k\} = ½$

These probabilities are not affected by the values of the RVs $S_1, S_2, \ldots S_{n-1}$ also the conditional probability of S_{n+1} given S_n depends on the value of S_n and not on the manner in which the value of S_n was reached.

Random process $\{X(t)\}$ (with Markov property) which take discrete values ,whether t is discrete or continuous, are called **Markov chains.**

Definition of Markov chain : If ,for all n,

$$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \ldots, X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\}$$

then the process $\{x_n\}$,n=0,1,2,.....is called a Markov Chain.

- ($a_{1,}$ $a_{2,\ldots.}$ a_n.....) are called the **states** of the **Markov** chain .
- The conditional probability

$$P\{X_n = a_j / X_{n-1} = a_i\}$$

is called the **one – step transition probability** from state a_i to state a_j at the n^{th} step (trial) and is denoted by P_{ij} (n-1,n).
- If the one – step transition probability does not depend on the step that is, P_{ij} (n-1,n)= P_{ij} (m-1,m) the Markov chain is called a **homogeneous Markov chain** or the chain is said to have **stationary transition probabilities**.

- when the Markov chain is homogeneous, the one - step transition probability is denoted by P_{ij}

  . The matrix $P=\{P_{ij}\}$ is called (one–step) **transition probability matrix,** shortly, tpm.
- NOTE: The tpm of a markov chain is a stochastic matrix, since $P_{ij} \geq 0$ and $\Sigma P_{ij} = 1$ for all i, i.e, the sum of all the elements of any row of the tpm is 1. This is obvious because the transition from state a_i to any one of the states (including a_i itself) is a certain event.
- The conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0, i.e,

$$P\{X_n = a_j / X_0 = a_i\}$$

  is called the **n – step transition probability** and denoted by $P_{ij}^{(n)}$.

An example in which how the tpm is formed for a Markov chain is explained .

Assume that a man is at an integral point of the X-axis between the origin and the point x=3. He takes a unit step either to the right with probability 0.7 or to the left with probability 0.3 , unless he is at the origin when he takes a step to the right to reach x=1 or he is at the point x=3 ,when he takes a step to the left to reach x=2.The chain is called ‘Random Walk with reflecting barriers’. The tpm is given below

$$
\begin{array}{llcccc}
 & & \multicolumn{4}{c}{\text{States of } X_n} \\
 & & 0 & 1 & 2 & 3 \\
\text{States of } X_{n-1} & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 0.3 & 0 & 0.7 & 0 \\
 & 2 & 0 & 0.3 & 0 & 0.7 \\
 & 3 & 0 & 0 & 1 & 0
\end{array}
$$

Definition: If the probability that the process is in state a_i is p_i (i=1,2,3,......k) at any arbitrary step, then the row vector $p=(p_{1,}\ p_2,....p_{.k})$ is called the **probability distribution of the process** at that time .

In particular , $p^{(0)} = \{p_1^{(1)}, p_2^{(2)}, ... p_k^{(k)}\}$

is the initial probability distribution.

Chapman-Kolmogorov Theorem

If p is the tpm of a homogeneous Markov chain, then the n-step tpm $p^{(n)}$ is equal to p^n .i.e.,

$[p_{ij}^{(n)}] = [p_{ij}]^{(n)}$

Definition :A stochastic matrix P is said to be

A regular matrix ,if all the entries of p^m (for some positive integer m) are positive . A homogeneous Markov chain is said to be regular if its tpm is regular.

Theorems:

1.If $p=\{p_i\}$is the state probability distribution of the process at an arbitrary ,time then that after one step is pP where P is the tpm of the chain and that after n steps in pP^n.

2. If a homogeneous Markov chain is regular ,then every sequence of state probability distributions approaches a unique fixed probability distribution called the stationary (state) distribution or steady –state distribution of the Markov chain. That is , $\lim_{n\to\infty}\{p^{(n)}\}=\pi,$

Where the state probability distribution at step n , Where the state probability distribution at step n ,

$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots p_k^{(n)})$ and the stationary distribution $\pi=(\pi_1, \pi_2, \ldots \pi_k)$ are row vectors.

3. Moreover ,if p is the tpm of the regular chain ,then $\pi P = \pi$ (π is a row vector).Using this property of π , it can be found out.

,

Classification of Markov Chain

- If $P_{ij}^{(n)} > 0$for some n and for all i and j ,every state can be reached from every other state .when this condition is satisfied ,the Markov chain is said to be **irreducible**
- The tpm of an irreducible chain is an **irreducible matrix** . Otherwise ,the chain is said to non -irreducible or **reducible**
- State i of a Markov Chain is called a **return state** ,if $P_{ii}^{(n)} > 0$ for some $n > 1$.
- The period di of a return state i is defined as the greatest common divisor of all m such that $P_{ii}^{(m)} > 0$,

  i.e ., $di = GCD\{m: P_{ii}^{(m)} > 0\}$

- State i is said to **periodic** with period d_i if $d_i > 1$ and **aperiodic** if $d_i = 1$. Obviously state i is aperiodic if $P_{ii} \neq 0$.
- The probability that the chain returns to state i, having started from state i ,for the first time at the n^{th} step (or after n transitions) is denoted by $f_{ii}^{(n)}$ and called the **first return time probability** or the **recurrence time probability**. $\{n, f_{ii}^{(n)}\}$, n= 1,2,3,...... is the distribution of recurrence times of the state i .
- If

$$F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$$

the return state i is **certain.**

$$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$$

Is called the **mean recurrence** time of the state i.

- A state i is said to be **persistent or Recurrent** if the return to state i is certain i. e., If $F_{ii}=1$.

- The state i is said to be **transient** if the return to state i is uncertain i.e., $F_{ii} < 1$.

- The state i is said to be **non -null persistent** if its mean recurrence time μ_{ii} is finite and null persistent ,if $\mu_{ii}= \infty$.

- A non- null persistent and aperiodic state is called **ergodic .**

Theorems without proof

The two theorems which will be helpful to classify the states of a Markov chain.

❖1. If a Markov chain is irreducible ,all its states are of the same type .they are all transient ,all null persistent or all non-null persistent . All its states are either aperiodic or periodic with the same period .

❖2. If a Markov chain is finite irreducible , all its states are non –null persistent.

Example -1

The transition probability matrix of a Markov chain $\{X_n\}$, n=1,2,3,.....having 3 states 1,2and 3 is

$$p=\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

and the initial distribution is $p^{(0)}$ =(0.7,0.2,0.1).Find (i) $P\{X_2 =3\}$ and (ii) $P\{X_3=2, X_2 =3, X_1 =3, X_0 =2\}$.

$$p^{(2)}= p^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & .0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & .0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(i)

$$P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3 / X_0 = i\} \times P\{X_0 = i\}$$

$$= p_{13}^{(2)} P(X_0 = 1) + P_{23}^{(2)} P(X_0 = 2) + p_{33}^{(2)} P(X_0 = 3)$$

$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1$$

$$= 0.182 + 0.068 + 0.029 = 0.279$$

(ii)

$$P\{X_1 = 3 / X_0 = 2\} = p_{23} = 0.2$$

$$P\{X_1 = 3, X_0 = 2\} = P\{X_1 = 3 / X_0 = 2\} \times P\{X_0 = 2\} = 0.2 \times 0.2 = 0.04$$

$$P\{X_2 = 3, X_1 = 3, X_0 = 2\} = P\{X_2 = 3 / X_1 = 3, X_0 = 2\} \times P\{X_1 = 3, X_0 = 2\}$$

$$= P\{X_2 = 3 / X_1 = 3\} \times P\{X_1 = 3, X_0 = 2\}$$

$$= 0.3 \times 0.04 = 0.012$$

$$P\{X_3 = 2, X_2 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$\times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= P\{X_3 = 2 / X_2 = 3\} \times P\{X_2 = 3, X_1 = 3, X_0 = 2\}$$

$$= 0.4 \times 0.012 = 0.0048$$

Example-2

A man either drives a car or catches a train to go to office each day . He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and drove to work if and only if a 6 appeared . Find (i)the probability that he takes a train on the third day and (ii)the probability that he drives to work in the long run.

Solution

The travel pattern is a Markov chain ,

with state space =(train ,car)

The tpm of the chain is

$$P = \begin{array}{c} \\ T \\ C \end{array}\begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

The initial state probability distribution is

$p^{(1)}$=(5/6, 1/6),since P (travelling by car)=P(getting 6 in the toss of the dice)=1/6

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

and P(travelling by train)=5/6

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

$\therefore$ P(the man travels by train on the third day)=11/24.

Let $\pi=(\pi_1, \pi_2)$be the limiting form of the state probability distribution or stationary state distribution of the Markov chain . By the property of π, $\pi p= \pi$, i,e.,

$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2),$$

$$i, e., \frac{1}{2}\pi_2 = \pi_1 \ldots\ldots(1)$$

$$\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \ldots\ldots(2)$$

$$\pi_1 + \pi_2 = 1$$

Equation (1) and (2) are one and the same.

Therefore ,consider (1) or (2) with $\pi_1 = \frac{1}{3} and \pi_2 = \frac{2}{3}$

Since π is a probability distribution .Solving ,

P(the man travels by car in the long run)=2/3.

Example -3

Three boys A,B, and C are throwing a ball to each other . A always throws the ball to B and B always throws the ball to C, but C is just as likely to throw the ball to B as to A . Show that the process is Markovian . Find the transition matrix and classify the states.

Solution:

Solution:

The transition probability matrix of the process $\{X_n\}$ is given by

$$\begin{array}{cc} & \text{State } X_n \\ \text{State } X_{n-1} \quad p = & \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array} \end{array}$$

States of X_n depend only on states of X_{n-1}, but not on the states of X_{n-2}, X_{n-3},...... or earlier states. Therefore, $\{X_n\}$ is a Markov chain.

Now

$$p^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}; p^3 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

and all other $p_{ij}^{(1)}$ >0. Therefore the chain is

$$p_{11}^{(3)} > 0, p_{13}^{(2)} > 0, p_{21}^{(2)} > 0, p_{22}^{(2)} > 0, p_{33}^{(2)} > 0,$$

irreducible.

Now

$$p^4 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}; p^5 = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{pmatrix}; p^6 = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 3/8 & 1/2 \\ 1/8 & 3/8 & 3/8 \end{pmatrix}$$

and so on. We note that $p_{ii}^{(2)}, p_{ii}^{(3)}, p_{ii}^{(5)}, p_{ii}^{(6)} ......are > 0$

for i=2,3 and GCD of 2,3,5,6…..=1.Therefore, the states 2 and 3 (i,e., B and C) are periodic with period 1,i.e., aperiodic.

We note that $p_{11}^{(3)}, p_{11}^{(5)}, p_{11}^{(6)} etc.are > 0$ and GCD of 3,5,6…..=1.

Therefore ,the state 1(i.e., state A) is periodic with period 1,i.e., aperiodic .Since the chain is finite and irreducible ,all its states are non-null persistent .Moreover all the states are ergodic.

Exercise(1)

1. Define a Markov process.
2. Define a Markov Chain and give an example.
3. When is a Markov Chain called homogeneous ?
4. Define transition probability matrix of a Markov chain.
5. Define n –step transition probability in a Markov chain.
6. State Chapman Kolmogorov theorem.
7. What is meant by steady state distribution of a Markov chain ?
8. Write down the relation satisfied by the steady state distribution and the tpm of a regular Markov chain.
9. If the tpm of a Markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ find the steady state distribution of the chain.

10. When is a Markov chain said to be irreducible or ergodic?
11. If the initial state probability distribution of a Markov chain is $p^{(0)}=(5/6,1/6)$ and the tpm of the chain is $\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ find the probability distribution of the chain after 2 steps.
12. What do you mean by an absorbing Markov chain?
13. What is a stochastic matrix?
14. When is a stochastic matrix said to be regular?
15. When is a Markov chain completely specified?

Exercise(2)

1. The tpm of a Markov Chain with three states 0,1,2 is

$$p = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

and the initial state distribution of the chain is $P\{X_0=i\}=1/3$, $i=0,1,2$.
Find (i) $P\{X_2=2\}$ and
(ii) $P\{X_3=1, X_2=2, X_1=1, X_0=2\}$.

2. A man is at an integral point on the X-axis between the origin and the point 3 .He takes a unit step to the right with probability 1/3 or to the left with probability 2/3 , unless he is at the origin , where he takes a step to the right to reach the point 2.What is the probability that (i) he is at the point 1 after 3 walks? And (ii) he is at the point 1 in the long run?

3. Suppose that the probability of a dry day state 0) following a rainy day(state1) is 1/3 and that the probability of a rainy day following a dry day is 1/2 .Given that May 1 is a dry day ,find the probability

   that (i) May 3 is also a dry day and (ii)May 5 is also a dry day.

4. A fair dice is tossed repeatedly. If X_n denotes the maximumof the numbers occuring in the first n tosses , find the transition probability matrix P of the Markov chain$\{X_n\}$. Find also P^2 and $P\{X_2=6\}$.

5. Find the nature of the states of the Markov chain with the tpm

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

Answers

Exercise(1)

8. If $\pi = (\pi_1, \pi_2, \ldots\ldots \pi_n)$ is the steady –state distribution of the Chain whose tpm is the n^{th} order square matrix P, then $\pi P = \pi$.

9. $\pi_1 = \frac{1}{3}$, $\pi_2 = \frac{2}{3}$

11. $p^{(1)} = (11/14\ ,\ 13/14)$.

12. A state i of a Markov chain is said to be an absorbing state if $p_{ii}=1$, i.e.,if it is impossible to leave it.

13. A square matrix ,in which the sum of all the elements of each row is 1,is called a stochastic matrix.

14. A stochastic matrix P is said to be regular if all the entries of P^m (for some positive integer m) are positive.

15. A Markov chain is completely specified when the initial probability distribution and the tpm are given.

Exercise(2)

1. 1/6, 3/64

2. $p = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}$

   (i) 22/27 (ii) 3/7

3. (i) 5/12 (ii)173/432

4.

$$p = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$p^2 = \frac{1}{36} \begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{pmatrix}$$

4. $P(X_2=6)=91/216$

5. The chain is finite and irreducible ,all its states are nonnull persistent . All states are not ergodic.

References

1. Veerarajan.T., Probability, Statistics and Random Processes.
2. Gross .D and Harris .C.M. Fundamentals of Queuing and Computer Science Applications.
3. Allen .A.O., Probability, Statistics and Queuing theory.