# Two Dimensional Random Variables

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**Definition:**

Let S be a sample space associated with a random experiment E. Let X and Y be two random variables defined on S. then the pair (X,Y) is called a Two – dimensional random variable.

The value of (X,Y) at a point $s \in S$ is given by the ordered pair of real numbers (X(s), Y(s)) = (x, y) where X(s) = x, Y(s) = y.

**Two – Dimensional discrete random variable:**

If the possible values of (X,Y) are finite or countably infinite, then (X,Y) is called a two-dimensional discrete random variable. When (X,Y) is a two-dimensional discrete random variable the possible values of (X,Y) may be represented as (xi, yj), i= 1, 2, 3, …n, j = 1, 2, 3, …m.

Example: 1

Consider the experiment of tossing a coin twice. The sample space is S = {HH, HT, TH, TT}.

Let X denotes the number of heads obtained in the first toss and Y denote the number of heads in the second toss. Then

| s | HH | HT | TH | TT |
|---|---|---|---|---|
| X(s) | 1 | 1 | 0 | 0 |
| Y(s) | 1 | 0 | 1 | 0 |

(X, Y) is a two-dimensional random variable or bi-variate random variable. The range space of (X, Y) is {(1,1), (1,0), (0,1), (0,0)} which is finite and so (X, Y) is a two-dimensional discrete random variables.

# Joint Probability Distribution

The probabilities of the two events $A = \{X \leq x\}\, and\, B = \{Y \leq y\}$ have defined as functions of x and y respectively called probability distribution functions.

$$F_x(x) = P(X \leq x)\, and\, F_y(y) = P(Y \leq y)$$

**Joint Probability Distribution of two random variables X and Y:**

The Joint Probability Distribution of two random variables X and Y is defined as $F_{X,Y}(x,y) = P\{X \leq x, Y \leq y\}$

**Properties of the joint distribution:**

A joint distribution function for the two random variables X and Y has several properties

$$1. F_{X,Y}(-\infty,-\infty) = 0; F_{X,Y}(-\infty, y) = 0; F_{X,Y}(x,-\infty) = 0$$

$$2. F_{X,Y}(\infty,\infty) = 1$$

$$3. 0 \leq F_{X,Y}(x,y) \leq 1$$

$$4. F_{X,Y}(x,y)\, is\, a\, non-decreasing\;\; function\, of\;\; x\, and\;\; y\, and\, so\, on..$$

# Joint probability function of Discrete R.V

For a Discrete R V ,

The joint probability function of X and Y is defined as :

1. $p(x, y) \geq 0$

$\sum_x \sum_y p(x, y) = 1.$

The Marginal probability function is defined as

$$p_X(x) = \sum_y p(x, y) \qquad p_Y(y) = \sum_x p(x, y)$$

- And the conditional probability function is defined as

$$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)} \qquad p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)}$$

# Independence

Definition: Independence

Two random variables $X$ and $Y$ are defined to be *independent* if

$$p(x, y) = p_X(x)\, p_Y(y) \qquad \text{if } X \text{ and } Y \text{ are discrete}$$

**Note :** $$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)} = \frac{p_X(x)\, p_Y(y)}{p_X(x)} = p_Y(y)$$

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)} = \frac{p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x)$$

Thus, in the case of independence

marginal distributions $\equiv$ conditional distributions

**Example: 2**

Consider the random variables $X$ and $Y$ with the joint probability mass function as presented in the following table

| X \ Y | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 0 | 0.25 | 0.1 | 0.15 | 0.5 |
| 1 | 0.14 | 0.35 | 0.01 | 0.5 |
| $p_X(x)$ | 0.39 | 0.45 | 0.16 | |

The marginal probabilities are as shown in the last column and the last row

$$p_{Y/X}(0/1) = \frac{p_{X,Y}(0,1)}{p_X(1)}$$

$$= \frac{0.14}{0.39}$$

Example: 2.A

The joint PMF of X and Y is given by

| Px,y(x,y) | y = 0 | y = 1 | y = 2 |
|---|---|---|---|
| x = 0 | 0.01 | 0 | 0 |
| x = 1 | 0.09 | 0.09 | 0 |
| x = 2 | 0 | 0 | 0.81 |

Find the marginal PMFs for the random variables X and Y

Solution:

Marginal PMF: $P_X(x)$ and $P_Y(y)$ by rewriting the matrix in the above Example and placing the row sums and column sums in the margins

| $P_{X,Y}(x,y)$ | $y=0$ | $y=1$ | $y=2$ | $P_X(x)$ |
|---|---|---|---|---|
| $x=0$ | 0.01 | 0 | 0 | 0.01 |
| $x=1$ | 0.09 | 0.09 | 0 | 0.18 |
| $x=2$ | 0 | 0 | 0.81 | 0.81 |
| $P_Y(y)$ | 0.10 | 0.09 | 0.81 | |

**Two – Dimensional continuous random variable:**

If (X,Y) can assume all values in a specified region R in XY plane (X,Y) is called a two-dimensional continuous random variable.

# Joint probability function

- For a Continuous RV, the joint probability function:

$$f(x,y) = \text{Pf}[X = x, Y = y]$$

- Marginal distributions

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

- Conditional distributions

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \qquad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

# Independence

<u>Definition</u>: Independence

Two random variables $X$ and $Y$ are defined to be *independent* if

$$p(x, y) = p_X(x)\, p_Y(y) \qquad \text{if } X \text{ and } Y \text{ are discrete}$$

$$f(x, y) = f_X(x)\, f_Y(y) \qquad \text{if } X \text{ and } Y \text{ are continuous}$$

**Note :** $$p_{Y|X}(y|x) = \frac{p(x, y)}{p_X(x)} = \frac{p_X(x)\, p_Y(y)}{p_X(x)} = p_Y(y)$$

$$p_{X|Y}(x|y) = \frac{p(x, y)}{p_Y(y)} = \frac{p_X(x)\, p_Y(y)}{p_Y(y)} = p_X(x)$$

Thus, in the case of independence

marginal distributions $\equiv$ conditional distributions

# The Multiplicative Rule for densities

if $X$ and $Y$ are discrete

$$p(x, y) = \begin{cases} p_X(x)\, p_{Y|X}(y|x) \\ p_Y(y)\, p_{X|Y}(x|y) \end{cases}$$

$$= p_X(x)\, p_Y(y) \quad \text{if } X \text{ and } Y \text{ are independent}$$

if $X$ and $Y$ are continuous

$$f(x, y) = \begin{cases} f_X(x)\, f_{Y|X}(y|x) \\ f_Y(y)\, f_{X|Y}(x|y) \end{cases}$$

$$= f_X(x)\, f_Y(y) \quad \text{if } X \text{ and } Y \text{ are independent}$$

**Example:** 3

For random variables X and Y, the joint probability density function is given by

$$f_{X,Y}(x,y) = \frac{1+xy}{4} \quad |x| \le 1, \ |y| \le 1$$

$$= 0 \quad \text{otherwise}$$

Find the marginal density $f_X(x), f_Y(y)$ and $f_{Y/X}(y/x)$. Are $X$ and $Y$ independent?

$$f_X(x) = \int_{-1}^{1} \frac{1+xy}{4} dy$$

$$= \frac{1}{2}$$

Similarly

$$f_Y(y) = \frac{1}{2} \qquad -1 \le y \le 1$$

and

$$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{1+xy}{2} \qquad \ne f_Y(y)$$

Hence, $X$ and $Y$ are not independent.

Example: 4

Let X and Y have the joint density $f(x,y)=\frac{6}{7}(x+y)^2, 0\le x\le 1, 0\le y\le 1$

By integrating over the appropriate regions, find

(i)P(X>Y),(ii)P(X+Y≤1),(iii)P(X≤½).

Sol:

$$f(x,y)=\frac{6}{7}(x+y)^2, 0\le x\le 1, 0\le y\le 1$$

(i) $P(X>Y)=\int_0^1\int_0^x \frac{6}{7}(x+y)^2\,dydx=\int_0^1 \frac{2}{7}(x+y)^3\Big|_{y=0}^{y=x} dx=\frac{1}{2}$

(ii) $P(X+Y\le 1)=\int_0^1\int_0^{1-x} \frac{6}{7}(x+y)^2\,dydx$

(iii) $P(X\le \frac{1}{2})=\int_0^1\int_0^{\frac{1}{2}} \frac{6}{7}(x+y)^2\,dxdy$

Example: 5 Let $f(x,y)=c(x^2-y^2)e^{-x},\ \ 0\le x<\infty,\ \ -x\le y<x$

(a)Find $c$. (b)Find the marginal densities.

Solution: (a)

$$f(x,y)=c(x^2-y^2)e^{-x},\quad 0\le x\le\infty,\quad -x\le y\le x$$

$$\int_0^\infty\int_{-x}^{x} c(x^2-y^2)e^{-x}\,dydx=\int_0^\infty\int_{-x}^{x} ce^{-x}(x^2-y^2)\,dydx$$

$$=\int_0^\infty ce^{-x}\left(x^2y-\frac{1}{3}y^3\right)\Bigg|_{y=-x}^{y=x}dx$$

$$=\frac{4c}{3}\int_0^\infty e^{-x}x^3\,dx$$

$$=\frac{4c}{3}\Gamma(4)$$

$$=\frac{-4c\cdot 3!}{3}$$

$$=1,\ \therefore\ c=\frac{1}{8}$$

$$(b).\ f_X(x) = \int_{-x}^{x} \frac{1}{8}(x^2 - y^2)e^{-x}dy = \frac{1}{6}x^3 e^{-x}, \quad x \ge 0$$

$$f_Y(y) = \begin{cases} \int_{y}^{\infty} f(x,y)dx, & y \ge 0 \\ \int_{-y}^{\infty} f(x,y)dx, & y < 0 \end{cases}$$

$$\therefore\ f_Y(y) = \begin{cases} \frac{1}{4}e^{y}(1+y), & y \ge 0 \\ \frac{1}{4}e^{y}(1-y), & y < 0 \end{cases}$$

Example: 6

Let $X$ and $Y$ have the joint densities function $f(x, y) = k(x - y)$, $0 \le y \le x \le 1$ and 0 elsewhere.

(a)Find $k$ . (b)Find the marginal densities of $X$ and $Y$ .

Solution:

(a)

$$f(x, y) = k(x - y) \quad , \quad 0 \le y \le x \le 1$$

$$\int_0^1 \int_0^x k(x - y)\,dydx$$

$$= \int_0^1 (kxy - \frac{1}{2}ky^2)\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 kx^2 - \frac{1}{2}kx^2\,dx$$

$$= \frac{k}{2}(\frac{x^3}{3})\Big|_0^1 = \frac{k}{6} = 1 , \therefore k = 6$$

$(b)$ *The* marginal densities of X and Y is

$$f_X(x) = \int_0^x 6(x - y)dy$$

$$f_Y(y) = \int_y^1 6(x - y)dx$$

Example: 7

A point is generated on a unit disk in the following way: The radius, $\boldsymbol{R}$, is uniform on $[\mathbf{0,1}]$, and the angle $\boldsymbol{\Theta}$ is uniform on $[\mathbf{0,2}\boldsymbol{\pi}]$ and is independent of $\boldsymbol{R}$.

(a) Find the joint density of $\boldsymbol{X} = \boldsymbol{R}\cos\boldsymbol{\Theta}$ and $\boldsymbol{Y} = \boldsymbol{R}\sin\boldsymbol{\Theta}$.

(b) Find the marginal densities of $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Solution:

**(a)**

$$R \sim \mathbf{U}[0,1]$$

$$\Theta \sim \mathbf{U}[0,2\pi]$$

$$\begin{cases} X = R\cos\Theta \\ Y = R\sin\Theta \end{cases} \Rightarrow \begin{cases} R = \sqrt{X^2+Y^2} \\ \Theta = \tan^{-1}\dfrac{Y}{X} \end{cases}$$

$$|J| = \left\| \begin{matrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{matrix} \right\|^{-1} = \frac{1}{\sqrt{x^2+y^2}}$$

$$f_{XY}(x,y) = f_{R\Theta}(\sqrt{x^2+y^2}, \tan^{-1}\frac{y}{x}) \cdot |J| = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{x^2+y^2}}, \quad x^2+y^2 \le 1$$

(b) $f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{XY}(x,y)dy$

$$= \frac{2}{2\pi} \int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{x^2+y^2}} dy$$

$$= \frac{2}{2\pi} [\ln(y + \sqrt{x^2+y^2})] \Big|_0^{\sqrt{1-x^2}}$$

$$= \frac{1}{\pi} \ln(\frac{\sqrt{1-x^2}+1}{|x|}), \quad -1 \le x \le 1$$

$$\left( \int \frac{1}{\sqrt{a^2+u^2}} du = \ln(u + \sqrt{a^2+u^2}) + c, \quad a > 0 \right)$$

**Similarly,** $f_Y(y) = \frac{1}{\pi} \ln(\frac{\sqrt{1-y^2}+1}{|y|}), \quad -1 \le y \le 1$

Example: 8

Suppose that $X$ and $Y$ have the joint density function

$$f(x,y) = c\sqrt{1-x^2-y^2}, \quad x^2+y^2 \le 1$$

Find the marginal densities of $X$ and $Y$.

Solution:

$$\iint_{x^2+y^2\le 1} c\cdot\sqrt{1-x^2-y^2}\,dxdy = c\int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\cdot r\cdot drd\theta$$

$$= c\int_0^{2\pi} \frac{-1}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^1 d\theta$$

$$= c\cdot\frac{1}{3}\cdot\int_0^{2\pi} d\theta$$

$$= \frac{2c\pi}{3}$$

$$= 1$$

$$\therefore c = \frac{3}{2\pi}$$

$$f_X(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy \qquad (let\ 1-x^2=a^2)$$

$$=\frac{3}{2\pi}\int_{-a}^{a}\sqrt{a^2-y^2}\,dy$$

$$=\frac{3}{2\pi}\left(\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right)\Bigg|_{-a}^{a}$$

$$=\frac{3}{4}(1-x^2)$$

$$\therefore f_X(x)=\frac{3}{4}(1-x^2),\ -1\le x\le 1$$

$$Similarly\ f_Y(y)=\frac{3}{4}(1-y^2),\ -1\le y\le 1$$

$$\because f(x,y)\ne f_X\cdot f_Y$$

$\therefore X$ **and** $Y$ **are not independent.**

Example: 9

$$f_{X,Y}(x,y)=\begin{cases} 4xy & 0\le x\le 1, 0\le y\le 1 \\ 0 & otherwise \end{cases}$$

Are X and Y independent?

Sol)

The marginal PDFs of X and Y are

$$f_X(x)=\begin{cases} 2x & 0\le x\le 1, \\ 0 & otherwise \end{cases} \qquad f_Y(x)=\begin{cases} 2y & 0\le y\le 1, \\ 0 & otherwise \end{cases}$$

It is easily verified that $f_{X,Y}(x,y)=f_X(x)f_Y(y)$ for all pairs (x,y) and so we conclude

that X and Y are independent

- ***Theorem:***

***Statement:***

Given two random variables $X$ and $Y$, the following equality is true:

$$E[X+Y] = E[X] + E[Y].$$

*Proof:*

Regarding $X + Y$ as a function of two random variables $g(X, Y)$, we can apply Proposition 7.1 to get

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)dydx \;+\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dxdy$$

$$= \int_{-\infty}^{\infty} x(\int_{-\infty}^{\infty} f(x,y)dy)dx \;+\; \int_{-\infty}^{\infty} y(\int_{-\infty}^{\infty} f(x,y)dx)dy$$

$$= \int_{-\infty}^{\infty} xf_X(x)dx \;+\; \int_{-\infty}^{\infty} yf_Y(y)dy$$

$$= E[X] + E[Y].$$

# Covariance between X & Y

- Covariance = 0 for independent X, Y
  - Positive for large X with large Y
  - Negative for large X with small Y (vice versa)
  - Formula is similar to our familiar variance formula

$$Cov[X,Y] = E(XY) - E(X) \bullet E(Y)$$

$$\rho(X,Y) = \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$Var(aX + bY + c) = a^2 Var[X] + b^2 Var[Y] + 2abCov[X,Y]$$

**Transformation of two dimensional random variables:**

Let $Z = g(X,Y)$, where g is the transformation function of X and Y, yielding a new random variable Z.

Standard transformations of this type are

$$(i)\, Z = X + Y, (ii)\, Z = XY, (iii)\, Z = \frac{X}{Y}, (iv)\, Z = \sqrt{X^2 + Y^2}$$

Many problems of type $Z = g(X,Y)$ can be solved by introducing an auxiliary variable $W = h(X,Y)$ and obtain the joint pdf of (Z, W). If the joint pdf of (X, Y) is $f_{XY}$, then the joint pdf of (Z, W), $f_{ZW}$ is given by

Where the Jocobian of transformation is

$$J = \frac{\partial(x,y)}{\partial(z,w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

The range space of (Z,W) is obtained from the range space of (X,Y) and the transformation, then the required pdf of Z is obtained as the marginal pdf of (Z,W) from the joint pdf $f_{ZW}$ by the following formula

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw$$

**Example: 10**

If X and Y are independent random variables each following normal with mean o S.D = 2. Find the pdf of Z = 2X + 3Y

**Solution:**

**Given** X and Y are independent normal random variables each following normal with mean o S.D = 2.

So, $\mu_X = 0, \sigma_X = 2, \mu_Y = 0, \sigma_Y = 2$

By the property of normal distribution

2X + 3Y is a normal variate with

$$\mu = 2\mu_X + 3\mu_Y = 2+3 = 0, and \sigma^2 = 2^2\sigma_X^2 + 3^2\sigma_Y^2 = 4.2^2 + 9.2^2 = 52$$

Z is a normal R.V with mean $\mu = 0 and S.D = \sqrt{52}$

So, the p.d.f of Z is $f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)dw = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$

$$= \frac{1}{\sqrt{52}\sqrt{2\pi}} e^{-\frac{z^2}{2\times 52}} \quad = \frac{1}{\sqrt{52}\sqrt{2\pi}} e^{-\frac{z^2}{104}}, -\infty < Z < \infty$$

***Central limit theorem:***

If $\bar{X}$ is the sample mean of a random sample $X_1$. $X_2$. …, $X_n$ of size $n$ arising from a population random variable $X$ with mean $\mu$ and variance $\sigma^2$, then for any ε > 0, the following equality is true:

$$\lim_{n\to\infty} P\{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \varepsilon\} = \Phi(\varepsilon)$$

where Φ(ε) is the cdf of the standard normal random variable (called the *error function* hereafter). That is, as $n \to \infty$, the random variable $Y = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is approximately a unit normal random variable.

## Examples

1) The mass of bars of chocolate produced by a factory have a normal distribution with mean 105g and standard deviation 4g. A random sample of 20 chocolate bars is chosen. What is the probability that the sample mean is less than 103g?

Let $X$ = mass of a bar of chocolate. Then $X \sim N[105, 16]$.

Let $\bar{X}$ be the mean weight of the sample. Then $\bar{X} \sim N\left[\mu, \frac{\sigma^2}{n}\right] = N\left[105, \frac{16}{20}\right] = N[105, 0.8]$

So $P(\bar{X} < 103) = P\left(Z < \frac{103-105}{\sqrt{0.8}}\right) = P(Z < -2.236)$

$= 1 - P(Z < 2.236) = 1 - 0.9873$

$= 0.0127.$

2) A random sample of size 100 is taken from B(20, 0.6). Find the probability that $\overline{X}$ is greater than 12.4.

- $X \sim B(20, 0.6)$. Therefore $\mu = 20 \times 0.6 = 12$ and $\sigma^2 = 20 \times 0.6 \times 0.4 = 4.8$.

As the sample size is large, $\overline{X} \approx N\left[\mu, \frac{\sigma^2}{n}\right] = N\left[12, \frac{4.8}{100}\right] = N[12, 0.048]$

So $P(\overline{X} > 12.4) = P\left(Z > \frac{12.4 - 12}{\sqrt{0.048}}\right) = P(Z > 1.826)$

$= 1 - P(Z < 1.826) = 1 - 0.9660 = 0.034$