

Unit IV: TRANSIENT ANALYSIS

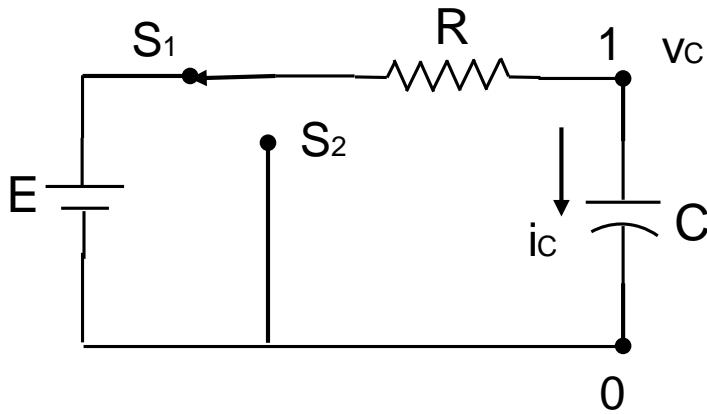
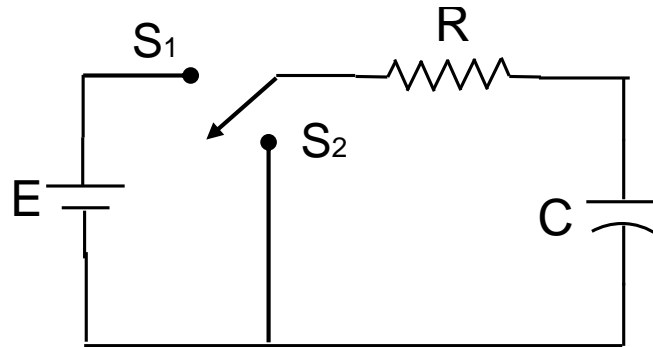
7.1 INTRODUCTION

So far steady state analysis of electric circuits was discussed. Electric circuits will be subjected to sudden changes which may be in the form of opening and closing of switches or sudden changes in sources etc. Whenever such a change occurs, the circuit which was in a particular steady state condition will go to another steady state condition. **Transient analysis is the analysis of the circuits during the time it changes from one steady state condition to another steady state condition.**

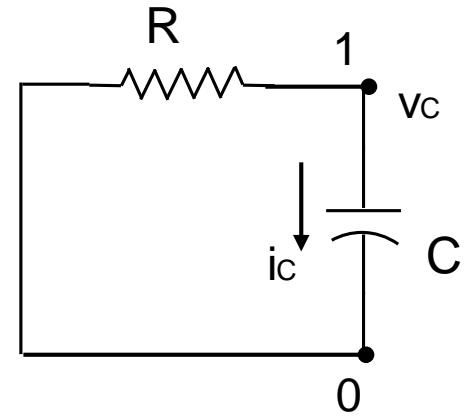
Transient analysis will reveal how the currents and voltages are changing during the transient period. **To get such time responses, the mathematical models should necessarily be a set of differential equations.** Setting up the mathematical models for transient analysis and obtaining the solutions are dealt with in this chapter.

A quick review on various test signals is presented first. Transient response of simple circuits using classical method of solving differential equations is then discussed. Laplace Transform is a very useful tool for solving differential equations. After introducing the Laplace Transform, its application in getting the transient analysis is also discussed.

What is TRANSIENT ANALYSIS?



With steady state condition, at time $t = 0$ switch position is changed from S_1 and S_2



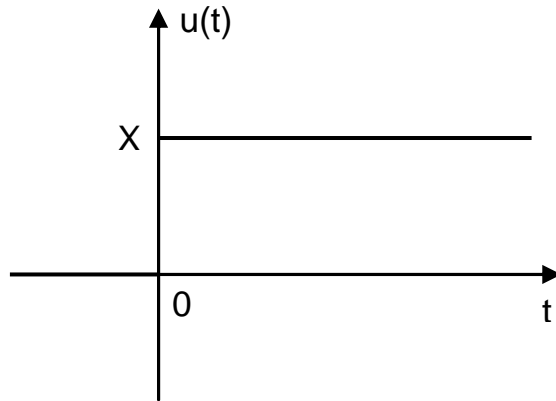
For $t \geq 0$, both v_C and i_C change with respect time.

Step function

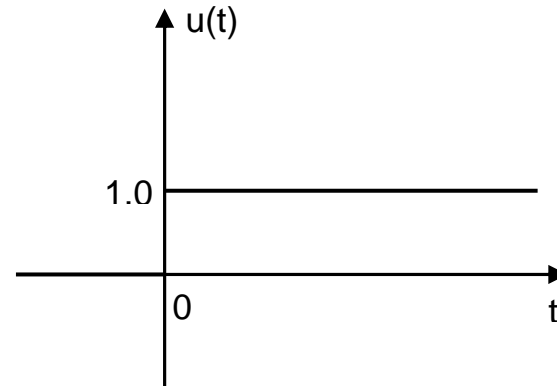
Step function is denoted as $u(t)$ and is described by

$$\left. \begin{aligned} u(t) &= X \text{ for } t \geq 0 \\ &= 0 \text{ for } t < 0 \end{aligned} \right\} \quad (7.2)$$

Fig. (a) shows a step function.



(a)



(b)

The step function with $X = 1$ is called as unit step function. It is described as

$$\left. \begin{aligned} u(t) &= 1.0 \text{ for } t \geq 0 \\ &= 0 \text{ for } t < 0 \end{aligned} \right\} \quad (7.3)$$

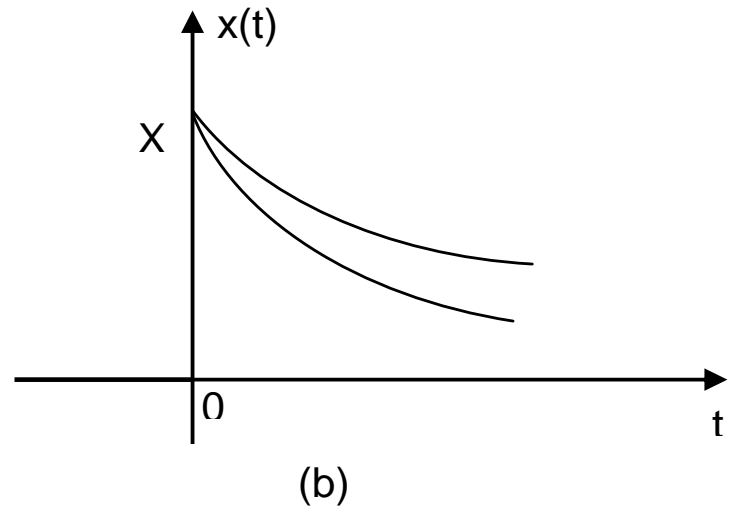
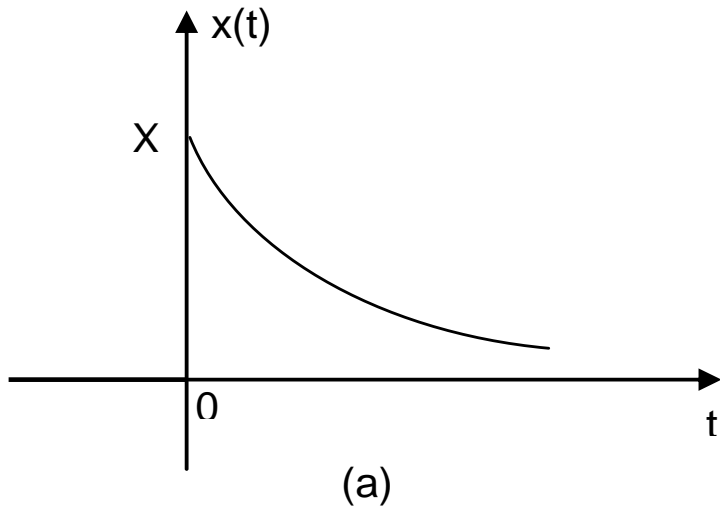
Unit step function is shown in Fig. (b).

Exponentially decaying function

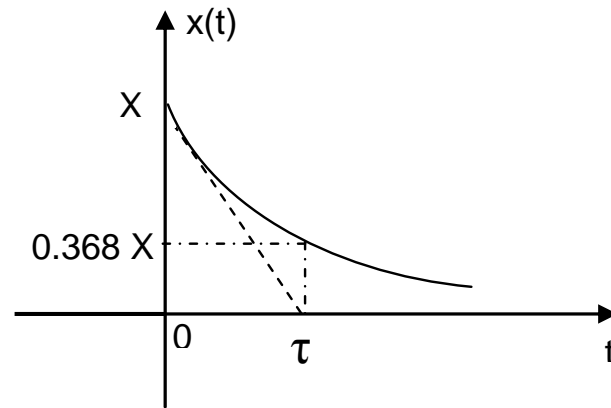
Exponentially decaying function is described by

$$\left. \begin{aligned} x(t) &= X e^{-\alpha t} \text{ for } t \geq 0 \\ &= 0 \text{ for } t < 0 \end{aligned} \right\} \quad (7.4)$$

The value of this function decreases exponentially with time as shown in Fig. below.



For exponentially decaying function, the time required for the signal to reach zero value, when it is decreased at a constant rate, equal to the rate of decay at time $t = 0$, is called **TIME CONSTANT**. Time constant is the measure of rate of decay.



Consider the exponentially decaying signal shown and described by

$$x(t) = X e^{-\alpha t} \quad (7.5)$$

Its slope at time $t = 0$ is given by

$$\left. \frac{dx}{dt} \right|_{t=0} = -\alpha X e^{-\alpha t} \Big|_{t=0} = -\alpha X \quad (7.6)$$

Minus sign indicates that the function value is decreasing with increase in time. Then,

as stated by the definition, time constant τ is given by

$$\tau = \frac{X}{\alpha X} = \frac{1}{\alpha} \quad (7.7)$$

For this exponentially decaying function, knowing $\alpha \tau = 1$, the value of $x(t)$ at time $t = \tau$ is obtained as

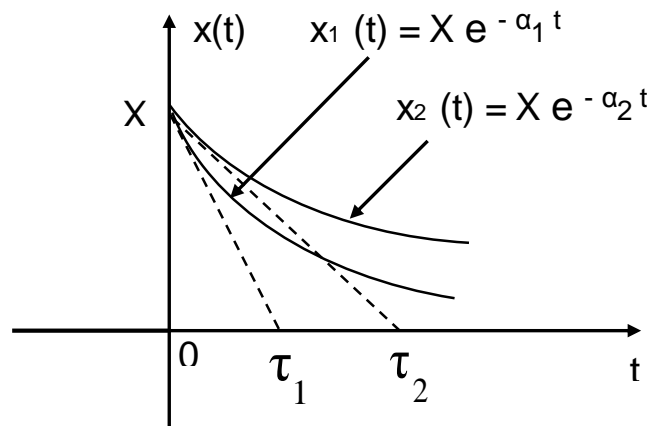
$$x(t) \Big|_{t=\tau} = X e^{-\alpha t} \Big|_{t=\tau} = X e^{-1} = 0.368 X$$

Therefore, for exponentially decaying function, time constant τ is also defined as the time required for the function to reach 36.8 % of its value at time $t = 0$. This aspect is shown in previous Fig.

Now consider the two exponentially decaying signals shown. They are described by

$$x_1(t) = X e^{-\alpha_1 t}$$

$$x_2(t) = X e^{-\alpha_2 t}$$

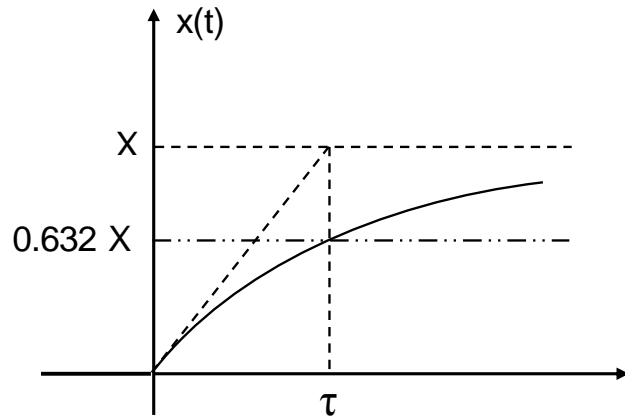


Their time constants are τ_1 and τ_2 respectively. It is seen that $\tau_1 < \tau_2$ and hence $\alpha_1 > \alpha_2$. Further, it can be noted that, smaller the time constant faster is the rate of decay.

Exponentially increasing function

The plot of $x(t) = X (1 - e^{-\alpha t})$ (7.36)

is shown in the Fig. It is to be seen that at time $t = 0$, the function value is zero and the function value tends to X as time t tends to ∞ . This is known as exponentially increasing function



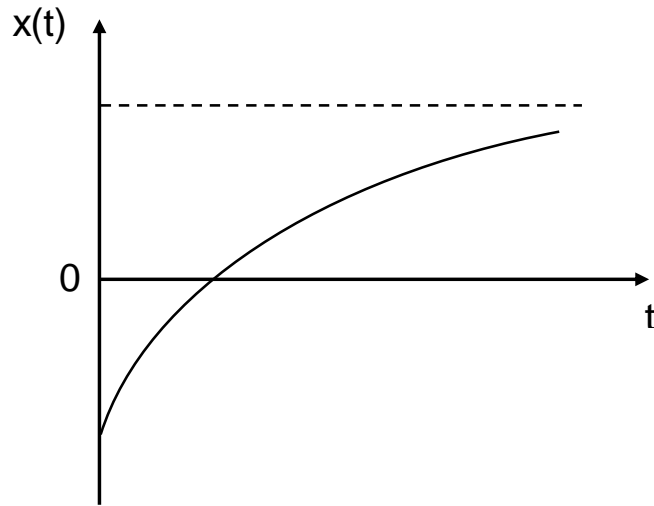
For such exponentially increasing function, time constant, τ is the time required for the function to reach the final value, if the function is increasing at the rate given at time $t = 0$.

$$\left. \frac{dx}{dt} \right|_{t=0} = 0 + \alpha X e^{-\alpha t} \Big|_{t=0} = \alpha X \quad \text{Therefore} \quad \tau = \frac{X}{\alpha X} = \frac{1}{\alpha} \quad (7.37)$$

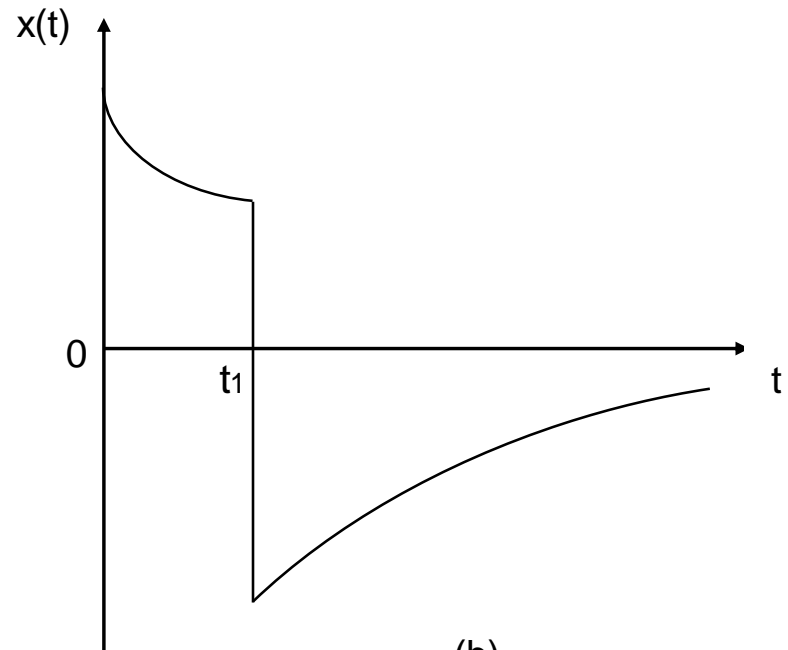
The value of $x(t)$ at time $t = \tau$ is obtained as $x(t) = X (1 - e^{-1}) = 0.632 X$ (7.38)

Thus, for exponentially increasing function, time constant τ is also defined as the time taken for the function to reach 63.2 % of the final value. This is shown in Fig. above.

In the Fig. (a) shown below, $x(t)$ is continuous.



(a)



(b)

In Fig. (b) shown, $x(t)$ has discontinuity at time $t = t_1$. The value of $\frac{dx}{dt}$ at time $t = t_1$ tends to infinity.

7.3 CERTAIN COMMON ASPECTS OF RC AND RL CIRCUITS

While doing transient analysis on simple RC and RL circuits, we need to make use of the following two facts.

1. **The voltage across a capacitor as well as the current in an inductor cannot have discontinuity.**
2. **With dc excitation, at steady state, capacitor will act as an open circuit and inductor will act as a short circuit.**

These two aspects can be explained as follows.

The current through a capacitor is given by $i_C = C (dv / dt)$. If the voltage across the capacitor has discontinuity, then at the time when the discontinuity occurs, dv / dt becomes infinity resulting the current i_C to become infinity. However, in physical system, we exclude the possibility of infinite current. Then, we state that in a capacitor, the voltage cannot have discontinuity. Suppose, if the circuit condition is changed at time $t = 0$, the capacitor voltage must be continuous at time $t = 0$ and hence $v_C(0^+) = v_C(0^-)$. (7.14)

where time 0^+ refers the time just after $t = 0$ and time 0^- refers the time just before $t = 0$.

Similarly the voltage across an inductor is $v_L = L (di / dt)$. If the current through the inductor has discontinuity, then at the time when the discontinuity occurs, di / dt becomes infinity resulting the voltage v_L to become infinity. However, in physical system, we exclude the possibility of infinite voltage. Then, we state that in an inductor, the current cannot have discontinuity. Suppose, if the circuit condition is changed at time $t = 0$, the inductor current must be continuous at time $t = 0$ and hence $i_L(0^+) = i_L(0^-)$ (7.15)

With dc excitation, at steady state condition, all the element currents and voltages are of dc in nature. Therefore, both di / dt and dv / dt will be zero. Since $i_C = C (dv / dt)$ and $v_L = L (di / dt)$, with dc excitation, at steady state condition, the current through the capacitor as well as the voltage across the inductor will be zero. In other words, with dc excitation, at steady state condition, the capacitor will act as an open circuit and the inductor will act as a short circuit.

Switching occurs at time $t = 0$

$$v_C(0^+) = v_C(0^-) \quad i_L(0^+) = i_L(0^-)$$

With DC excitation, at steady state

capacitor acts as OPEN CIRCUIT and

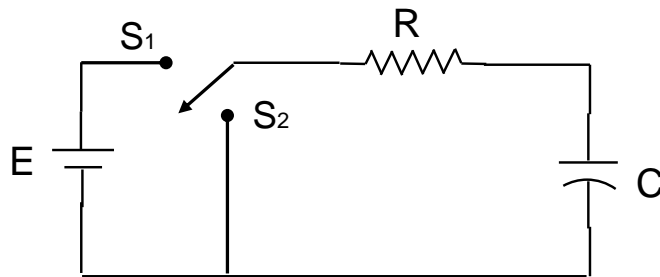
inductor acts as SHORT CIRCUIT

7.4 TRANSIENT IN RC CIRCUIT

While studying the transient analysis of RC and RL circuits, we shall encounter with two types of circuits namely, **source free circuit** and **driven circuit**.

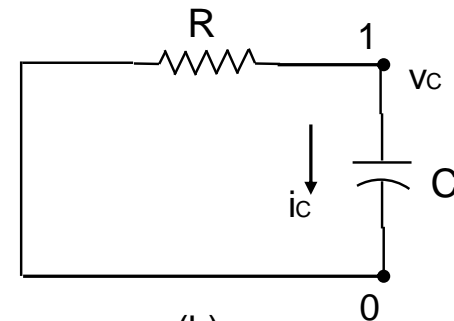
Source free circuit

A circuit that does not contain any source is called a source free circuit. Consider the circuit shown in Fig. 7.7 (a). **Let us assume that the circuit was in steady state condition with the switch is in position S_1 for a long time.** Now, the capacitor is charged to voltage E and will act as open circuit.



(a)

Fig. 7.7 Source free RC circuit.



(b)

Suddenly, at time $t = 0$, the switch is moved to position S_2 . The voltage across the capacitor and the current through the capacitor are designated as v_c and i_c respectively. The voltage across the capacitor will be continuous. Hence

$$v_c(0^+) = v_c(0^-) = E$$

(7.16)

The circuit for time $t > 0$ is shown in Fig. 7.7 (b). We are interested in finding the voltage across the capacitor as a function of time. Later, if required, current through the capacitor can be calculated from $i_C = C \frac{dv}{dt}$. Voltage at node 1 is the capacitor voltage v_C . The node equation for the node 1 is

$$\frac{v_C}{R} + C \frac{dv_C}{dt} = 0 \quad (7.17)$$

$$\text{i.e. } \frac{dv_C}{dt} + \frac{v_C}{RC} = 0 \quad (7.18)$$

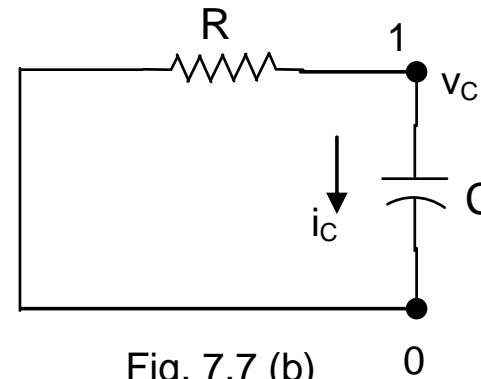


Fig. 7.7 (b)

We have to solve this first order differential equation (DE) with the initial condition

$$v_C(0^+) = E \quad (7.19)$$

We notice that DE in Eq. (7.18) is a **homogeneous equation** and hence will have **only complementary solution**. Let us try $v_C(t) = K e^{st}$ (7.20)

as a possible solution of Eq. (7.18).

$$\frac{dv_c}{dt} + \frac{v_c}{RC} = 0 \text{ with the initial condition } v_c(0^+) = E$$

A possible solution is: $v_c(t) = K e^{st}$

Substituting the solution in the DE. we get

$$s K e^{st} + \frac{1}{RC} K e^{st} = 0 \quad \text{i.e. } K e^{st} \left(s + \frac{1}{RC} \right) = 0$$

The above equation will be satisfied if

$$K e^{st} = 0 \text{ and or } \left(s + \frac{1}{RC} \right) = 0$$

From Eq. (7.20) it can be seen that $K e^{st} = 0$ will lead to the trivial solution of $v_c(t) = 0$.

We are looking for the non-trivial solution of Eq. (7.18). Therefore

$$s + \frac{1}{RC} = 0 \tag{7.21}$$

$$s + \frac{1}{RC} = 0 \quad (7.21)$$

This is the characteristic equation of the DE given in Eq. (7.18). Its solution $s = -\frac{1}{RC}$ is called the root of the characteristic equation. It is also called as the natural frequency because it characterizes the response of the circuit in the absence of any external source. Thus the solution of the DE (7.18) is obtained by substituting $s = -\frac{1}{RC}$ in the solution $v_C(t) = K e^{st}$. Therefore,

$$v_C(t) = K e^{-\frac{1}{RC} t} \quad (7.22)$$

The constant K can be found out by using the initial condition of $v_C(0) = E$. Substituting $t = 0$ in the above equation, we get

$$v_C(0) = K = E \quad (7.23)$$

Thus the solution is

$$v_C(t) = E e^{-\frac{1}{RC} t} \quad (7.24)$$

Thus the solution is
$$v_C(t) = E e^{-\frac{1}{RC} t} \quad (7.24)$$

It can be checked that this solution satisfy

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0 \text{ with the initial condition } v_C(0^+) = E$$

Obtained solution is sketched in Fig. 7.8. It is an exponentially decaying function.

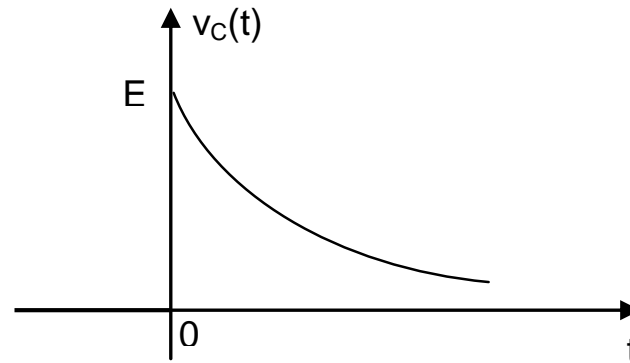


Fig. 7.8 Plot of $v_C(t)$ as given by Equation (7.24).

In this case, the time constant $\tau = RC$. By varying values of R and C , we can get different exponentially decaying function for $v_C(t)$. The dimension of time constant RC can be verified as time as shown below.

$$RC = \frac{\text{volt}}{\text{amp.}} \cdot \frac{\text{coulomb}}{\text{volt}} = \frac{\text{amp. sec.}}{\text{amp.}} = \text{sec.}$$

$$v_C(t) = E e^{-\frac{1}{RC}t} \quad (7.24)$$

The current through the capacitor, in the direction as shown in Fig. 7.7 (b), is given by

$$\begin{aligned} i_C(t) &= C \frac{dv_C}{dt} = CE \left(-\frac{1}{RC}\right) e^{-\frac{1}{RC}t} \\ &= -\frac{E}{R} e^{-\frac{1}{RC}t} \end{aligned} \quad (7.25)$$

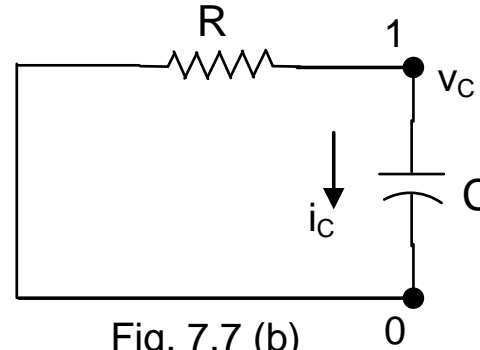


Fig. 7.7 (b)

Since the capacitor is discharging, the current is negative in the direction shown in Fig. 7.7 (b). The plot of capacitor current $i_C(t)$ is shown in Fig. 7.9.

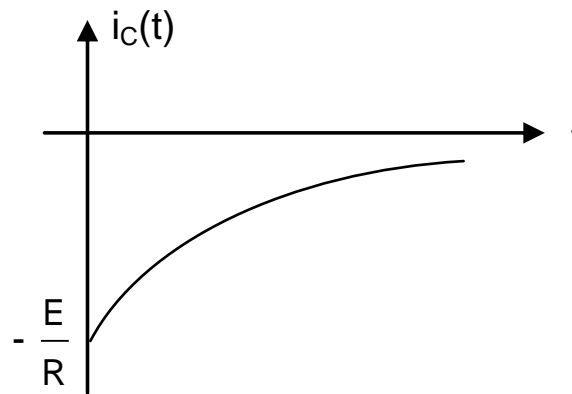


Fig. 7.9 Plot of $i_C(t)$ as given by Equation (7.25).

Driven circuit

Again consider the circuit shown in Fig. 7.7 (a) which is reproduced in Fig. 7.10 (a). Let us say that the switch was in position S_2 long enough so that $v_c(t) = 0$ and $i_c(t) = 0$ i.e. all the energy in the capacitor is dissipated and the circuit is at rest. Now, the switch is moved to position S_1 . We shall measure time from this instant. As discussed earlier, since the capacitor voltage cannot have discontinuity,

$$v_c(0^+) = v_c(0^-) = 0 \quad (7.26)$$

The circuit applicable for time $t > 0$, is shown in Fig. 7.10 (b).

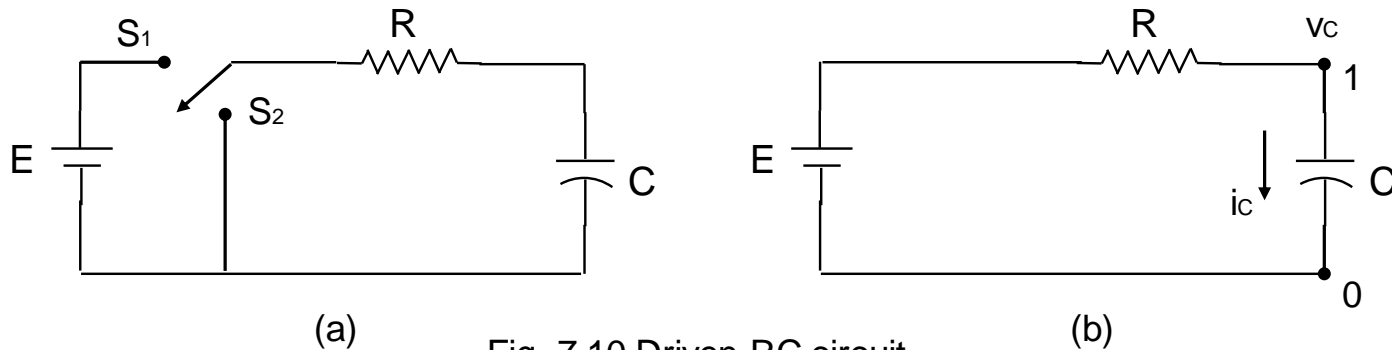


Fig. 7.10 Driven RC circuit.

Node equation for the node 1 gives

$$\frac{v_c - E}{R} + C \frac{dv_c}{dt} = 0 \quad (7.27)$$

$$\text{i.e. } \frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{E}{RC} \quad (7.28)$$

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{E}{RC} \quad (7.28)$$

Unlike in the previous case, now the right hand side is not zero, but contains a term commonly called the forcing function. For this reason, this circuit is classified as driven circuit. The initial condition for the above DE is

$$v_C(0^+) = 0 \quad (7.29)$$

The complete solution is given by

$$v_C(t) = v_{cs}(t) + v_{ps}(t) \quad (7.30)$$

where $v_{cs}(t)$ is the complementary solution and $v_{ps}(t)$ is the particular solution.

The complementary solution $v_{cs}(t)$ is the solution of the homogeneous equation

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = 0 \quad (7.31)$$

Recalling that Eq. (7.22) is the solution of Eq. (7.18), we get

$$v_{cs}(t) = K e^{-\frac{1}{RC} t} \quad (7.32)$$

Since the forcing function is a constant, the particular solution can be taken as

$$v_{ps}(t) = A$$

Since it satisfies the non-homogeneous equation given by Eq. (7.28), $\frac{dv_c}{dt} + \frac{v_c}{RC} = \frac{E}{RC}$

on substitution, we get

$$0 + \frac{A}{RC} = \frac{E}{RC} \text{ i.e. } A = E.$$

$$\text{Thus } v_{ps}(t) = E \quad (7.33)$$

$$\text{Addition of } v_{cs}(t) \text{ and } v_{ps}(t) \text{ yields } v_c(t) = K e^{-\frac{1}{RC}t} + E \quad (7.34)$$

To determine the value of K, apply the initial condition of $v_c(0) = 0$ to the above equation. Thus

$$0 = K + E \quad \text{i.e. } K = -E$$

$$\text{Thus, the complete solution is } v_c(t) = -E e^{-\frac{1}{RC}t} + E = E (1 - e^{-\frac{1}{RC}t}) \quad (7.35)$$

The plot of capacitor voltage $v_C(t) = E (1 - e^{-\frac{1}{RC}t})$ is shown in Fig. 7.11.

For this function, time constant τ is $= RC$.

The current through the capacitor is calculated as

$$i_C(t) = C \frac{dv_C}{dt} = C \frac{E}{RC} e^{-\frac{1}{RC}t}$$

$$= \frac{E}{R} e^{-\frac{1}{RC}t}$$

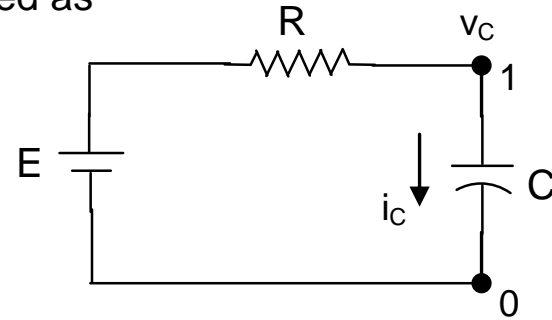


Fig. 7.10 (b)

(7.39)

Now, the capacitor current as marked in Fig. 7.10 (b), is positive and the capacitor gets charged. This capacitor current is plotted as shown in Fig. 7.12.

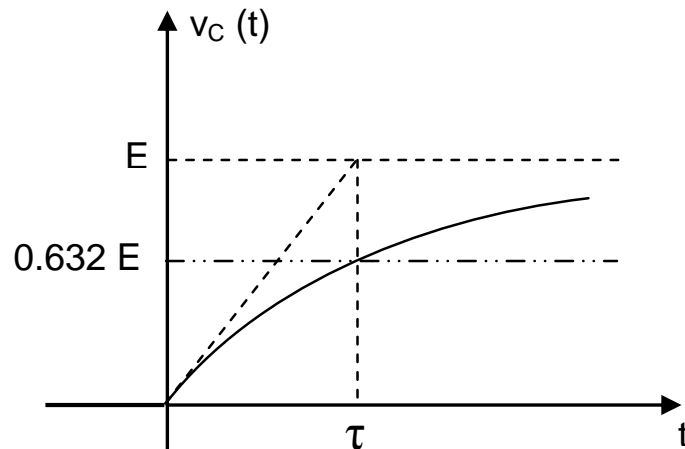


Fig. 7.11 Plot of $v_C(t)$ as given by Eqn. (7.35).

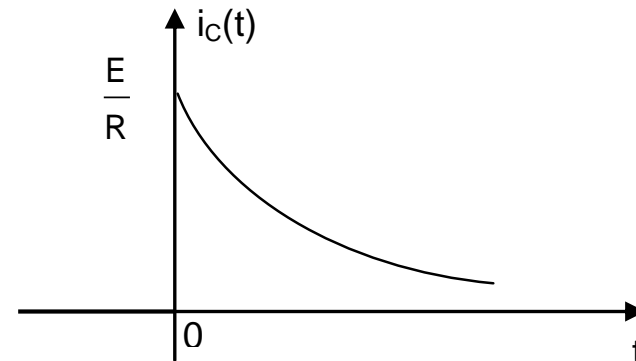


Fig. 7.12 Plot of $i_C(t)$ as given by Eqn. (7.39).

We have solved the circuits shown in Fig. 7.10 (b) and the resulting solutions are shown in Figs. 7.11 and 7.12. They are reproduced in Fig. 7.13.

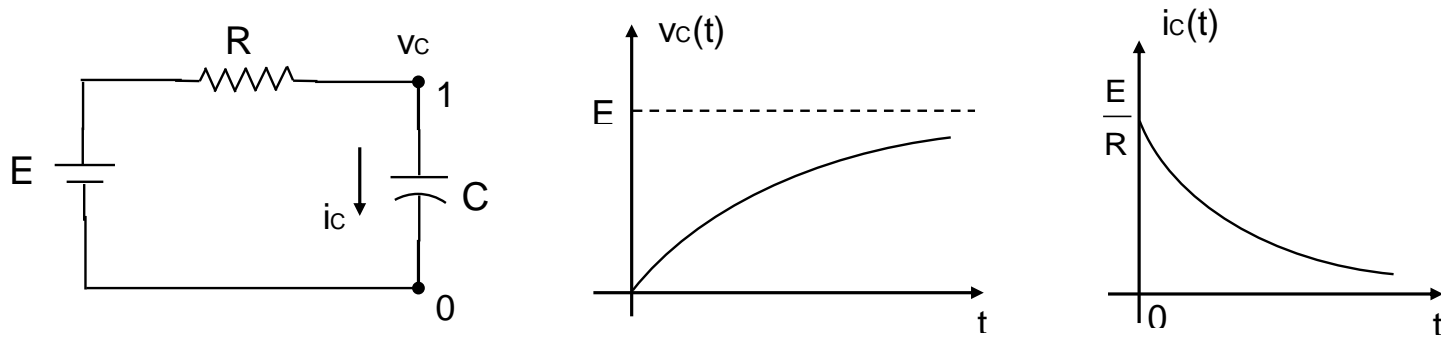


Fig. 7.13 RC driven circuit and voltage and current responses.

These results can be obtained straight away recognizing the following facts.

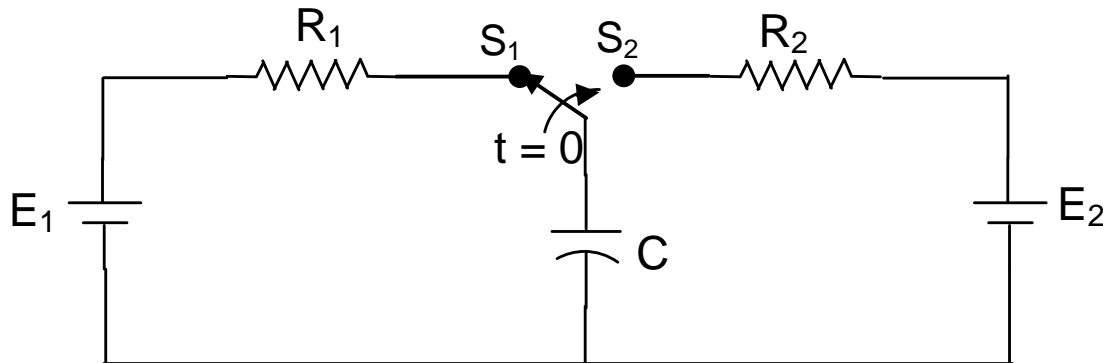
The solution of first order differential equation will be either exponentially decreasing or exponentially increasing. It is known that $v_c(0^+) = 0$. With dc excitation, at steady state, the capacitor will act as open circuit and hence $v_c(\infty) = E$. Thus, the capacitor voltage exponentially increases from 0 to E.

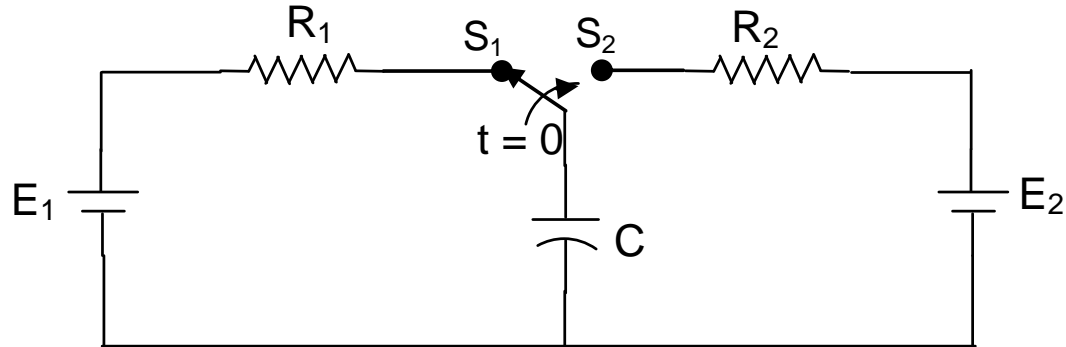
Since $v_c(0^+) = 0$, initially the capacitor is short circuited and hence $i_c(0) = \frac{E}{R}$. With dc excitation, at steady state, the capacitor will act as open circuit and hence $i_c(\infty) = 0$. Thus the capacitor current exponentially decreases from $\frac{E}{R}$ to zero.

Similar reasoning out is possible, in other cases also, to obtain the responses directly.

More general case of finding the capacitor voltage

In the previous discussion, it was assumed that the initial capacitor voltage $v_C(0) = 0$. There may be very many situations wherein initial capacitor voltage is not zero. There may be initial charge in the capacitor resulting non-zero initial capacitor voltage (Example 7.8). Further, the circuit arrangements can also cause non-zero initial capacitor voltage. For this purpose consider the circuit shown below. The switch was in position S_1 for a long time. It is moved from position S_1 to S_2 at time $t = 0$.





We shall assume the following:

1. At time $t = 0^-$ the circuit was at steady state condition with the switch in position S_1
2. After switching to position S_2 , the circuit is allowed to reach the steady state condition

Thus, we are interested about the transient analysis for one switching period only.

Initial capacitor voltage $v_C(0)$ is E_1 and the final capacitor voltage $v_C(\infty)$, will be E_2 .

The more general expression for the capacitor voltage can be obtained as

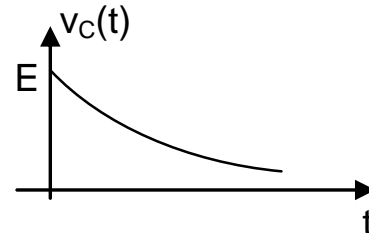
$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)] e^{-\frac{1}{R_2 C} t} \quad (7.47)$$

Summary of formulae useful for transient analysis on RC circuits

1. Time constant $\tau = RC$ $\alpha = 1 / RC$

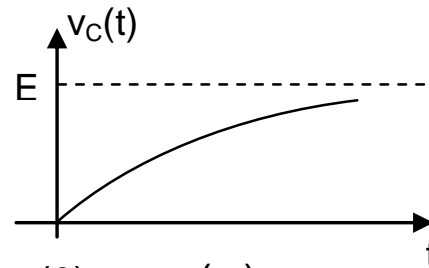
2. When the capacitor is discharging from the initial voltage of E

$$v_C(t) = E e^{-\frac{1}{RC} t}$$



3. When the capacitor is charged from zero initial voltage to final voltage of E

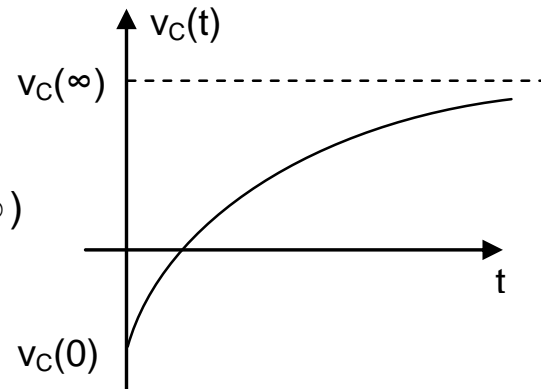
$$v_C(t) = E (1 - e^{-\frac{1}{RC} t})$$



4. When the capacitor voltage changes from $v_C(0)$ to $v_C(\infty)$

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)] e^{-\frac{1}{RC} t}$$

Plot of $v_C(t)$ depends on values of $v_C(0)$ and $v_C(\infty)$



5. Capacitor current $i_C(t) = C \frac{dv_C(t)}{dt}$

Example 7.1 An RC circuit has $R = 20 \, \Omega$ and $C = 400 \, \mu\text{F}$. What is its time constant?

Solution For RC circuit, time constant $\tau = RC$.

Therefore, $\tau = 20 \times 400 \times 10^{-6} \, \text{s} = 8 \, \text{ms}$

Example 7.2 A capacitor in an RC circuit with $R = 25 \, \Omega$ and $C = 50 \, \mu\text{F}$ is being charged with initial zero voltage. What is the time taken for the capacitor voltage to reach 40 % of its steady state value?

Solution With $R = 25 \, \Omega$ and $C = 50 \, \mu\text{F}$, $\tau = RC = 1.25 \times 10^{-3} \, \text{s}$; hence $1/RC = 800 \, \text{s}^{-1}$.

Taking the capacitor steady state voltage as E , $v_C(t) = E (1 - e^{-\frac{1}{RC} t})$

Let t_1 be the time at which the capacitor voltage becomes $0.4 E$. Then

$$0.4 E = E (1 - e^{-800 t_1}) \text{ i.e. } 0.4 = 1 - e^{-800 t_1}$$

$$e^{-800 t_1} = 0.6 \text{ i.e. } -800 t_1 = \ln 0.6 = -0.5108$$

$$\text{Therefore, } t_1 = \frac{0.5108}{800} \, \text{s} = 0.6385 \times 10^{-3} \, \text{s} = 0.6385 \, \text{ms}$$

Example 7.3 In an RC circuit, having a time constant of 2.5 ms, the capacitor discharges with initial voltage of 80 V. (a) Find the time at which the capacitor voltage reaches 55 V, 30 V and 10 V (b) Calculate the capacitor voltage at time 1.2 ms, 3 ms and 8 ms.

Solution (a) Time constant $RC = 2.5 \text{ ms}$; Thus $\frac{1}{RC} = \frac{1000}{2.5} = 400 \text{ s}^{-1}$

During discharge, capacitor voltage is given by $v_C(t) = 80 e^{-400 t} \text{ V}$

Let t_1 , t_2 and t_3 be the time at which capacitor voltage becomes 55 V, 30 V and 10 V.

$$55 = 80 e^{-400 t_1}; -400 t_1 = \ln \frac{55}{80} = -0.3747; \text{ Thus } t_1 = 0.93765 \text{ ms}$$

$$30 = 80 e^{-400 t_2}; -400 t_2 = \ln \frac{30}{80} = -0.9808; \text{ Thus } t_2 = 2.452 \text{ ms}$$

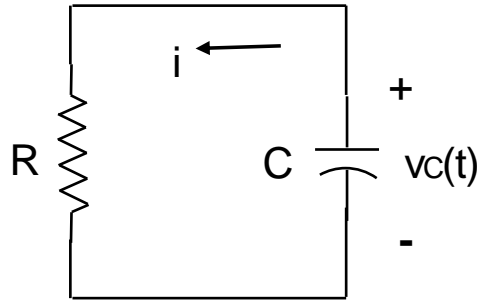
$$10 = 80 e^{-400 t_3}; -400 t_3 = \ln \frac{10}{80} = -2.0794; \text{ Thus } t_3 = 5.1985 \text{ ms}$$

$$(b) \quad v_C(1.2 \times 10^{-3}) = 80 e^{-400 t} = 80 e^{-0.48} = 49.5027 \text{ V}$$

$$v_C(3 \times 10^{-3}) = 80 e^{-400 t} = 80 e^{-1.2} = 24.0955 \text{ V}$$

$$v_C(8 \times 10^{-3}) = 80 e^{-400 t} = 80 e^{-3.2} = 3.261 \text{ V}$$

Example 7.4 Consider the circuit shown below.



Given

$$v_c(t) = 56 e^{-250t} \text{ V for } t > 0$$

$$i(t) = 7 e^{-250t} \text{ mA for } t > 0$$

- (a) Find the values of R and C. (b) Determine the time constant.
- (c) At what time the voltage $v_c(t)$ will reach half of its initial value?

Solution (a) Given that $v_c(t) = 56 e^{-250t} \text{ V}$. Therefore $\tau = RC = \frac{1}{250} \text{ s}$

$$\text{Resistance } R = \frac{v_c(t)}{i(t)} = 8000 \Omega; \text{ Thus capacitance } C = \frac{1}{250 \times 8000} \text{ F} = 0.5 \mu\text{F}$$

(b) Time constant $= RC = 4 \times 10^{-3} \text{ s} = 4 \text{ ms}$

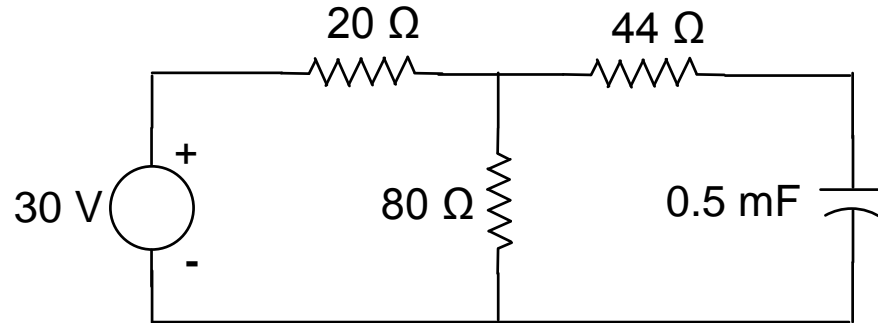
(c) Let t_1 be the time taken for the voltage to reach half of its initial value of 56 V.

$$\text{Then, } 56 e^{-250t_1} = 28; \text{ i.e. } e^{-250t_1} = 0.5 \quad \text{i.e. } -250t_1 = \ln 0.5 = -0.6931;$$

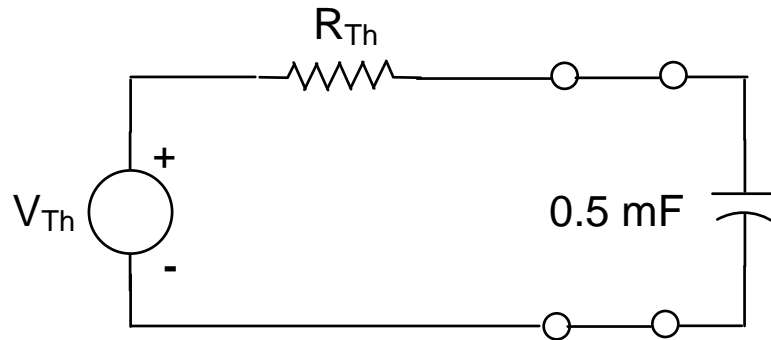
$$\text{Time } t_1 = \frac{0.6931}{250} \text{ s} = 2.7724 \times 10^{-3} \text{ s} = 2.7724 \text{ ms}$$

Example 7.5

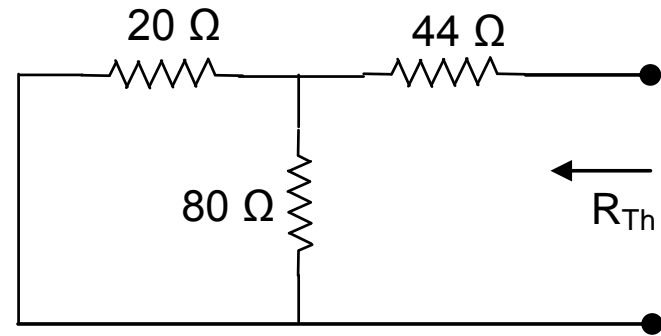
Find the time constant of the RC circuit shown in below.



Solution Thevenin's equivalent across the capacitor, is shown below.



(a)

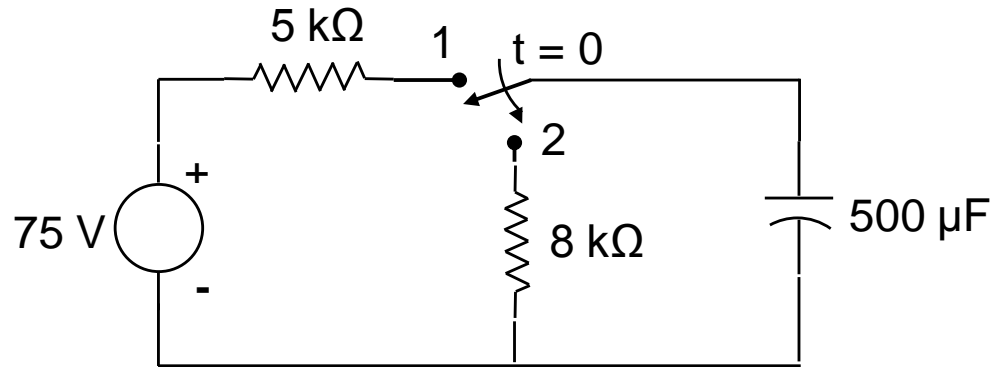


(b)

Referring to Fig. (b) above, $R_{Th} = 44 + (20 \parallel 80) = 60 \Omega$

Time constant $\tau = RC = 60 \times 0.5 \times 10^{-3} \text{ s} = 30 \text{ ms}$

Example 7.6 The switch in circuit shown was in position 1 for a long time. It is moved from position 1 to position 2 at time $t = 0$. Sketch the wave form of $v_c(t)$ for $t > 0$.



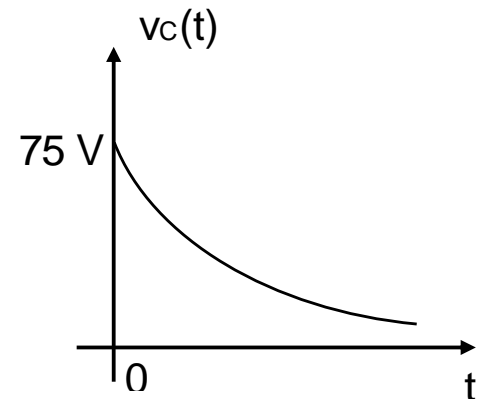
Solution With switch is in position 1, capacitor gets charged to a voltage of 75 V. i.e. $v_c(0^+) = 75 \text{ V}$. The switch is moved to position 2 at time $t = 0$.

Time constant $RC = 8 \times 10^3 \times 500 \times 10^{-6} = 4 \text{ s}$

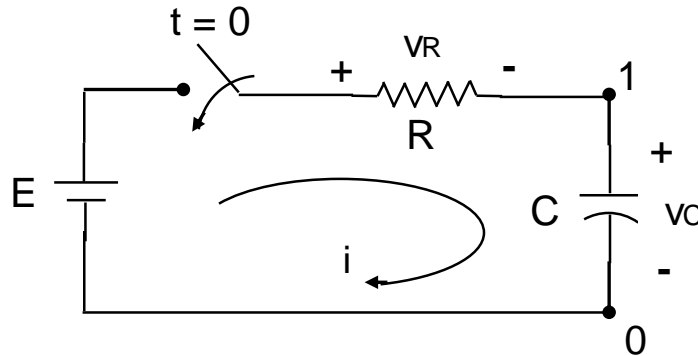
Finally the capacitor voltage decays to zero. Thus,

$$v_c(t) = 75 e^{-0.25 t}$$

Wave form of the capacitor voltage is shown.



Example 7.7 A series RC circuit has a constant voltage of E , applied at time $t = 0$ as shown in Fig. below. The capacitor has no initial charge. Find the equations for i , v_R and v_C . Sketch the wave shapes.



$$E = 100 \text{ V}$$

$$R = 5000 \Omega$$

$$C = 20 \mu\text{F}$$

Solution Since there is no initial charge, $v_C(0^+) = v_C(0^-) = 0$

For $t > 0$, capacitor is charged to final voltage of 100 V.

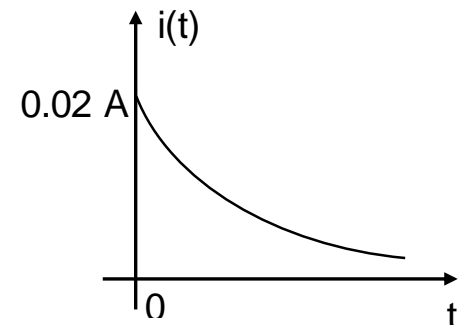
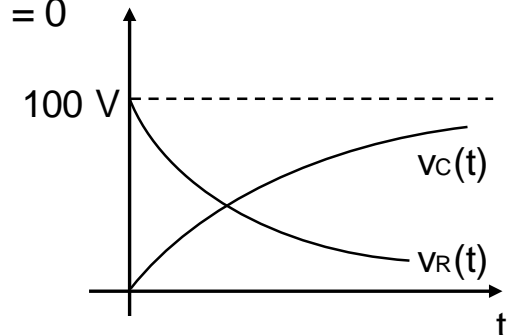
Time constant $RC = 5000 \times 20 \times 10^{-6} = 0.1 \text{ sec.}$

$$v_C(t) = E (1 - e^{-\frac{1}{RC}t}). \text{ Thus, } v_C(t) = 100 (1 - e^{-10t}) \text{ V}$$

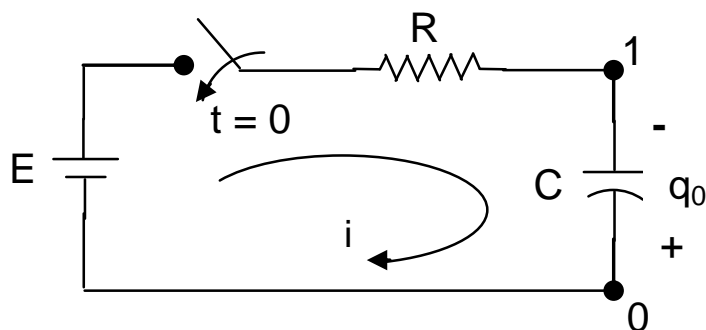
$$i(t) = C \frac{dv_C}{dt} = 20 \times 10^{-6} \times 100 \times 10 e^{-10t} = 0.02 e^{-10t} \text{ A}$$

$$\text{Voltage across the resistor is } v_R(t) = R i(t) = 100 e^{-10t} \text{ V}$$

Wave shapes of i , $v_R(t)$ and $v_C(t)$ are shown.



Example 7.8 A $20\ \mu\text{F}$ capacitor in the RC circuit shown has an initial charge of $q_0 = 500\ \mu\text{C}$ with the polarity as shown. The switch is closed at time $t = 0$. Find the current transient and the voltage across the capacitor. Find the time at which the capacitor voltage is zero. Also sketch their wave shape.



$$E = 50\ \text{V}$$

$$R = 1000\ \Omega$$

$$C = 20\ \mu\text{F}$$

Solution Initial charge of q_0 in the capacitor is equivalent to initial voltage of

$$v_C(0) = - \frac{q_0}{C} = - \frac{500 \times 10^{-6}}{20 \times 10^{-6}} = -25\ \text{V};$$

$$\text{Further, } v_C(\infty) = E = 50\ \text{V}$$

$$\text{Time constant } RC = 1000 \times 20 \times 10^{-6} = 20 \times 10^{-3}\ \text{s.} \quad \text{Thus } 1/RC = 50\ \text{s}^{-1}$$

$$v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)] e^{-\frac{1}{RC}t}$$

$$v_C(t) = 50 + [-25 - 50] e^{-50t} = 50 - 75 e^{-50t}$$

$$\text{Current } i(t) = C \frac{dv_C}{dt} = 20 \times 10^{-6} \times 75 \times 50 e^{-50t}\ \text{A} = 0.075 e^{-50t}\ \text{A}$$

Let t_1 be the time at which the capacitor voltage becomes zero. Then

$$50 - 75 e^{-50 t_1} = 0 \quad \text{i.e.} \quad e^{-50 t_1} = 0.6667$$

$$-50 t_1 = -0.4054 \quad \text{i.e.} \quad t_1 = 8.108 \times 10^{-3} \text{ s}$$

The capacitor voltage becomes zero at time $t_1 = 8.108 \text{ ms}$

Wave forms are shown in Fig. 7.24

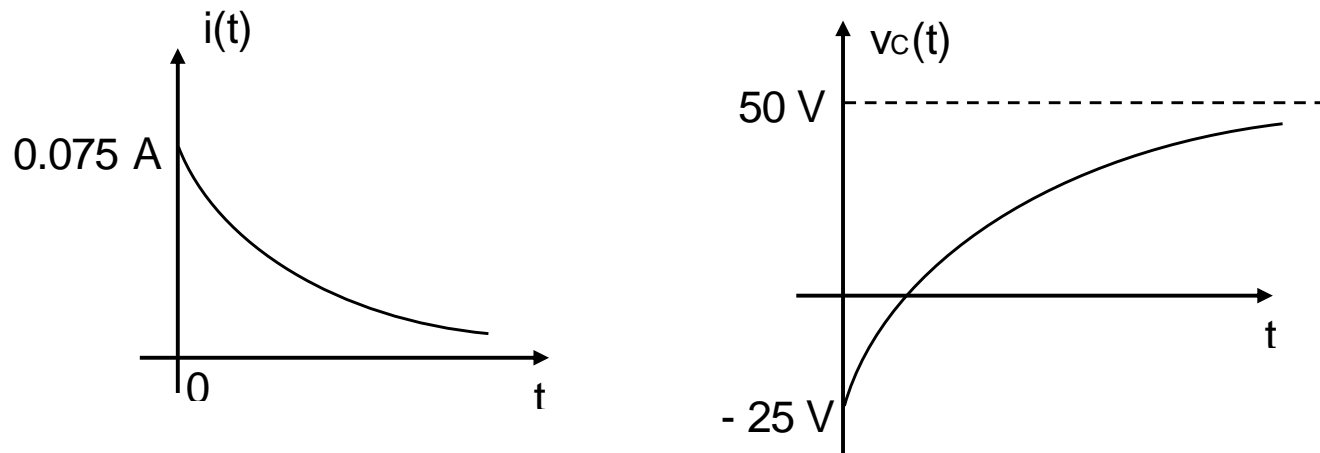
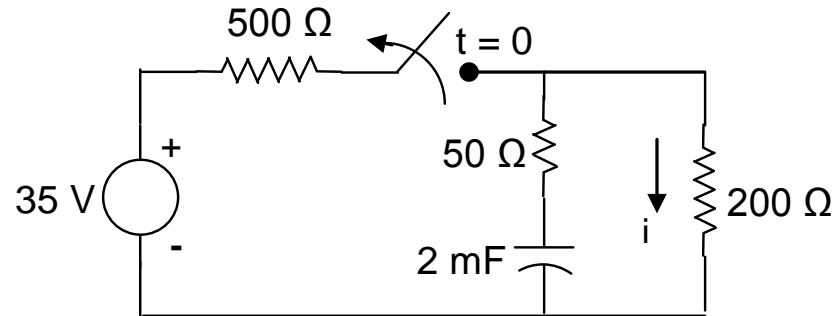


Fig. 7.24 Wave forms - Example 7.8.

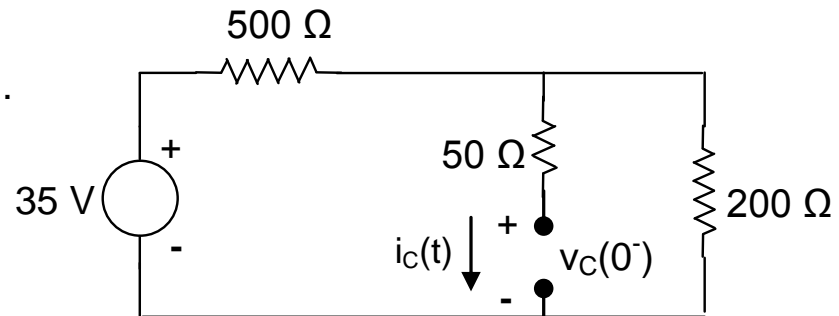
Example 7.9

Consider the circuit shown below. The switch was in closed position for a long time. It is opened at time $t = 0$. Find the current $i(t)$ for $t > 0$.



Solution Circuit at time $t = 0^-$ is shown.

$$v_C(0^-) = 35 \times \frac{200}{200 + 500} = 10 \text{ V}$$



For time $t > 0$, capacitor voltage of 10 V is discharged through a resistor of 250 Ω .

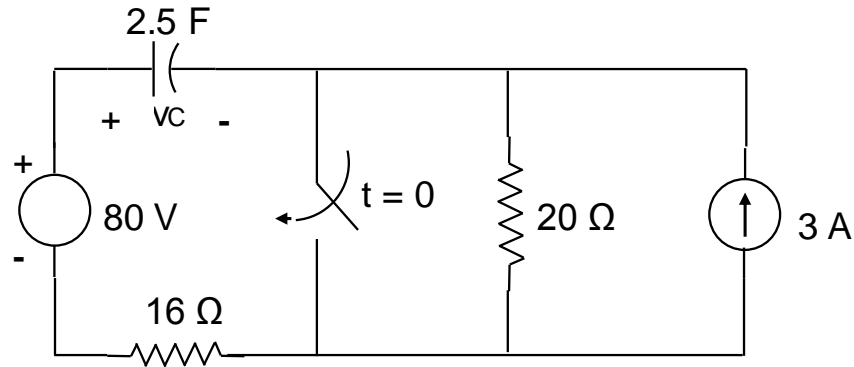
Time constant $RC = 250 \times 2 \times 10^{-3} = 0.5 \text{ s}$;

$$v_C(t) = 10 e^{-2t} \text{ V}$$

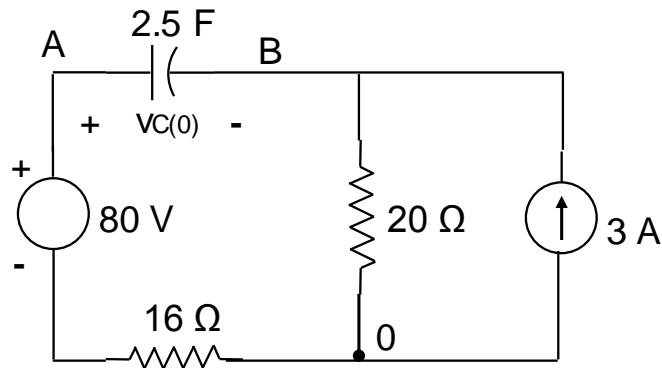
$$i_C(t) = C \frac{dv_C}{dt} = 2 \times 10^{-3} \times (-20) e^{-2t} \text{ A} = -40 \times 10^{-3} e^{-2t} \text{ A} = -0.04 e^{-2t} \text{ A}$$

$$\text{Thus } i(t) = -i_C(t) = 0.04 e^{-2t} \text{ A}$$

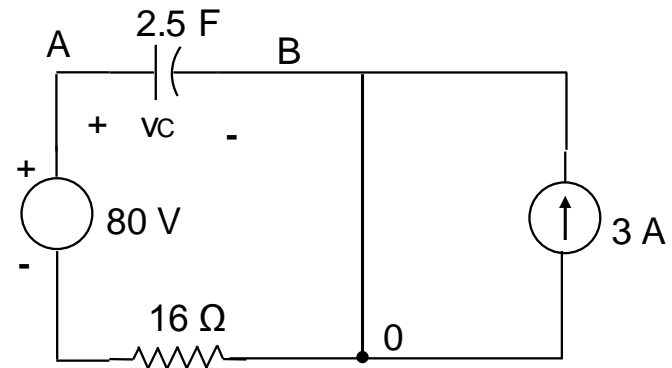
Example 7.10 Consider the circuit shown. The switch was in open position for a long time. It is operated as shown. Compute and plot the capacitor voltage for $t > 0$. Also find the time at which the capacitor voltage is 50 V.



Solution Circuit at time $t = 0^-$ is shown in Fig. (a).



(a)



(b)

Capacitor acts as open circuit. $I_{16\Omega} = 0$. Voltage $V_A = 80$ V and voltage $V_B = 60$ V

Thus $v_C(0) = 20$ V

With the switch is in closed position, the circuit will be as shown in Fig. (b). With the steady state reached, Capacitor acts as open circuit. $I_{16\ \Omega} = 0$.

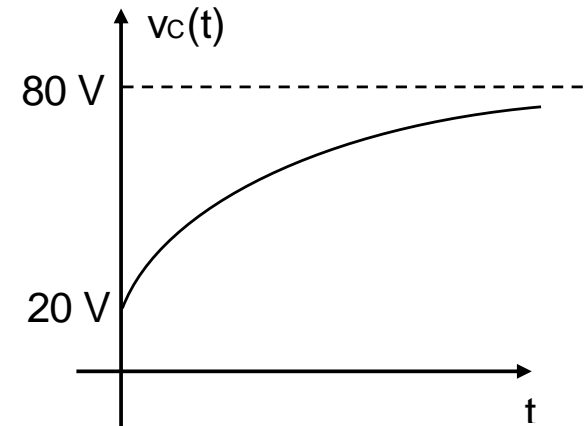
Voltage $V_A = 80\text{ V}$ and voltage $V_B = 0\text{ V}$. Thus $v_C(\infty) = 80\text{ V}$

$$RC = 16 \times 2.5 = 40\text{ s}$$

Using $v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)] e^{-\frac{1}{RC}t}$ we get

$$v_C(t) = 80 + [20 - 80] e^{-0.025t} = 80 - 60 e^{-0.025t}\text{ V}$$

Plot of the capacitor voltage is shown.



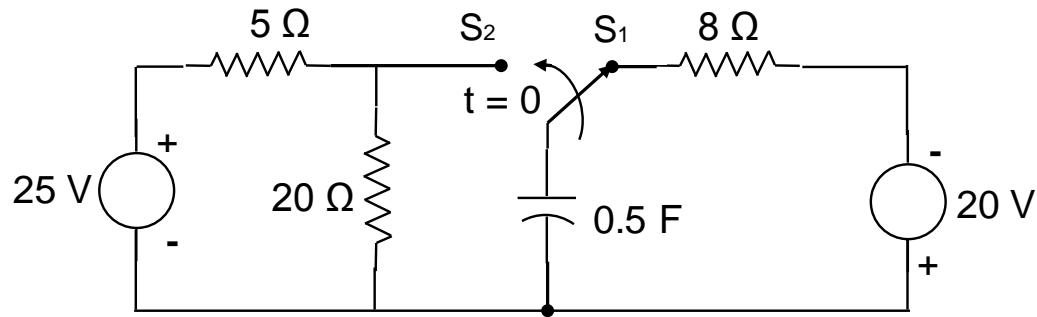
Let t_1 be the time at which the capacitor voltage = 50 V. Then

$$80 - 60 e^{-0.025t_1} = 50 \text{ i.e. } 60 e^{-0.025t_1} = 30 \text{ i.e. } e^{-0.025t_1} = 0.5 \text{ i.e. } -0.025 t_1 = -0.6932$$

$$\text{Thus } t_1 = 27.728\text{ s}$$

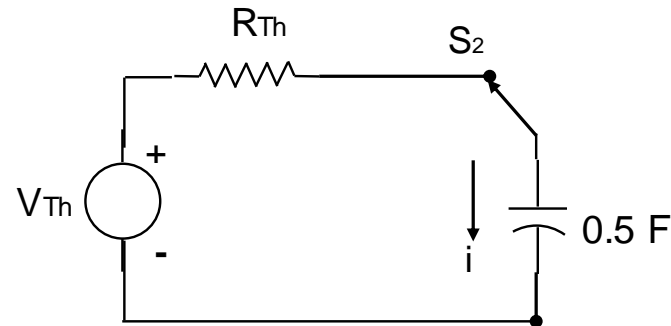
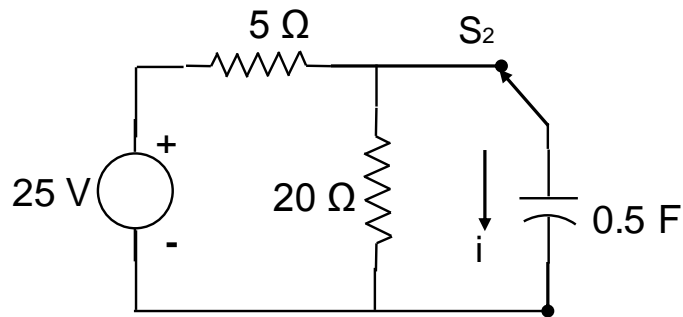
Capacitor voltage becomes 50 V at time $t_1 = 27.728\text{ s}$

Example 7.11 Consider the circuit shown below. The switch was in position S_1 for a long time. It is operated as shown. Compute and plot the capacitor voltage for $t > 0$. Also find the time at which the capacitor voltage becomes zero.



Solution Voltage $v_c(0) = -20\text{ V}$

Circuit for time $t > 0$ and its Thevenin's equivalent are shown below.



$$V_{Th} = \frac{20}{20 + 5} \times 25 = 20\text{ V}$$

$$R_{Th} = 5 \parallel 20 = 4\ \Omega; \text{ Thus } RC = 4 \times 0.5 = 2\text{ s}$$

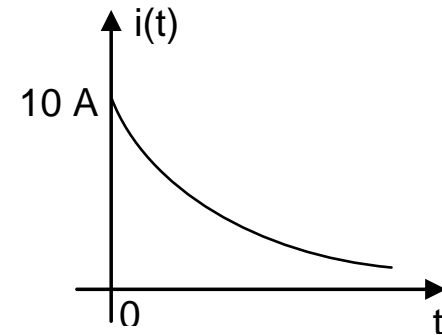
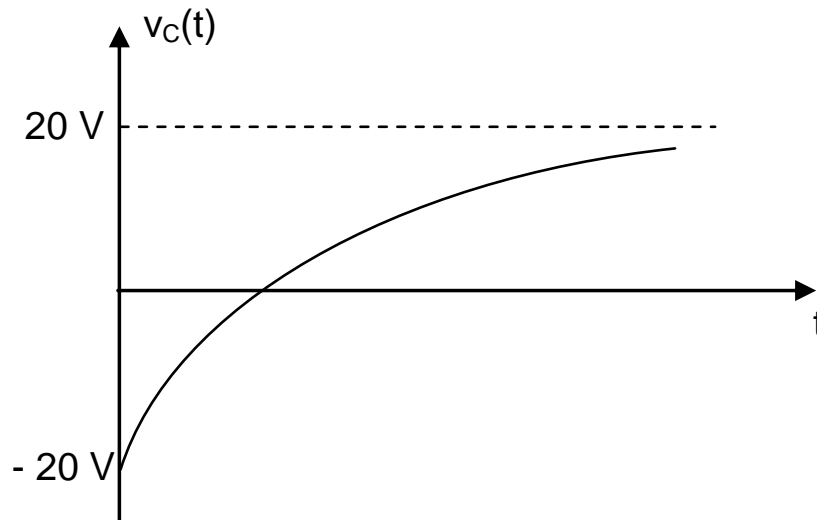
Using $v_c(t) = v_c(\infty) + [v_c(0) - v_c(\infty)] e^{-\frac{1}{RC}t}$ we get

$$v_c(t) = 20 + [-20 - 20] e^{-0.5t} = 20 - 40 e^{-0.5t}\text{ V}$$

$$v_C(t) = 20 - 40 e^{-0.5t} \text{ V}$$

$$i_C(t) = C \frac{dv_C}{dt} = 0.5 \times 20 e^{-0.5t} \text{ A} = 10 e^{-0.5t} \text{ A}$$

Wave shapes of $v_C(t)$ and $i_C(t)$ are shown below.



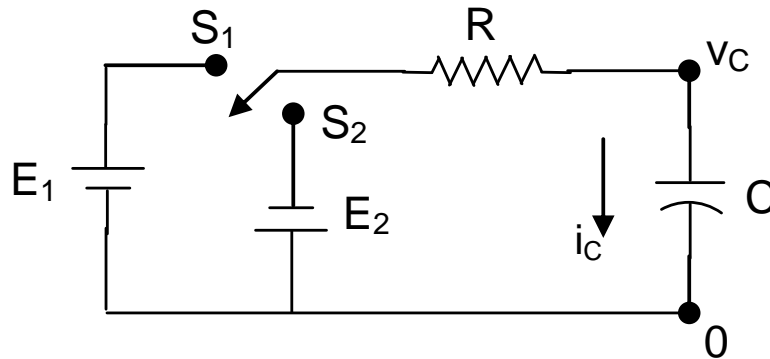
Let t_1 be the time at which the capacitor voltage reaches zero value. Then

$$20 - 40 e^{-0.5t_1} = 0; \text{ i.e. } e^{-0.5t_1} = 0.5; \text{ i.e. } -0.5 t_1 = -0.6931; \text{ Thus } t_1 = 1.3863 \text{ s}$$

Capacitor voltage reaches zero value at time $t_1 = 1.3863 \text{ s}$

So far we have done transient analysis for one switching period. Now we shall illustrate how to carry out transient analysis for two switching period through an example.

Example 7.12 In the initially relaxed RC circuit shown the switch is closed on to position S_1 at time $t = 0$. After one time constant, the switch is moved on to position S_2 . Find the complete capacitor voltage and current transients and show their wave forms.



$$E_1 = 20\text{ V}; E_2 = 40\text{ V}$$

$$R = 500\ \Omega$$

$$C = 0.5\ \mu\text{F}$$

Solution $RC = 500 \times 0.5 \times 10^{-6}\text{ s} = 0.25 \times 10^{-3}\text{ s} = 0.25\text{ ms}$ $1/RC = 4000\text{ s}^{-1}$

During the first switching period, capacitor gets charged from zero volt. Its voltage exponentially increases towards 20 V. Thus

$$v_C(t) = 20 (1 - e^{-4000 t})\text{ V}$$

$$\text{At } t = 1 \text{ time constant, } v_C = 20 (1 - e^{-1}) = 12.64\text{ V}$$

For the second switching operation, there is initial capacitor voltage of 12.64 V.

Let the second switching occurs at time $t' = 0$. Time $t' = 0$ implies time $t = 0.25 \times 10^{-3}$ s i.e. $t' = t - 0.25 \times 10^{-3}$. For $t' > 0$, capacitor voltage changes from its initial value, $v_c(0)$, of 12.64 V to final value, $v_c(\infty)$, of -40 V. Knowing that

$$v_c(t) = v_c(\infty) + [v_c(0) - v_c(\infty)] e^{-\frac{1}{RC}t} \text{ we get}$$

$$v_c(t') = -40 + [12.64 + 40] e^{-4000t'} = 52.64 e^{-4000t'} - 40 \text{ V}$$

Therefore, capacitor voltages for the two switching periods are

$$v_c(t) = 20 (1 - e^{-4000t}) \text{ V for } t > 0 \text{ and } \leq 0.00025 \text{ s}$$

$$v_c(t) = 52.64 e^{-4000(t - 0.00025)} - 40 \text{ V for } t \geq 0.00025 \text{ s}$$

$$\text{with } v_c(0.00025^-) = v_c(0.00025^+) = 12.64 \text{ V}$$

(Note that the capacitor voltage shall maintain continuity)

Knowing that

$$v_C(t) = 20 (1 - e^{-4000 t}) \text{ V for } t > 0 \text{ and } \leq 0.00025 \text{ s}$$

$$v_C(t) = 52.64 e^{-4000 (t - 0.00025)} - 40 \text{ V for } t \geq 0.00025 \text{ s}$$

For the first switching period

$$\text{Capacitor current } i_C(t) = C \frac{dv_C}{dt} = 0.5 \times 10^{-6} \times 20 \times 4000 e^{-4000 t} = 0.04 e^{-4000 t} \text{ A}$$

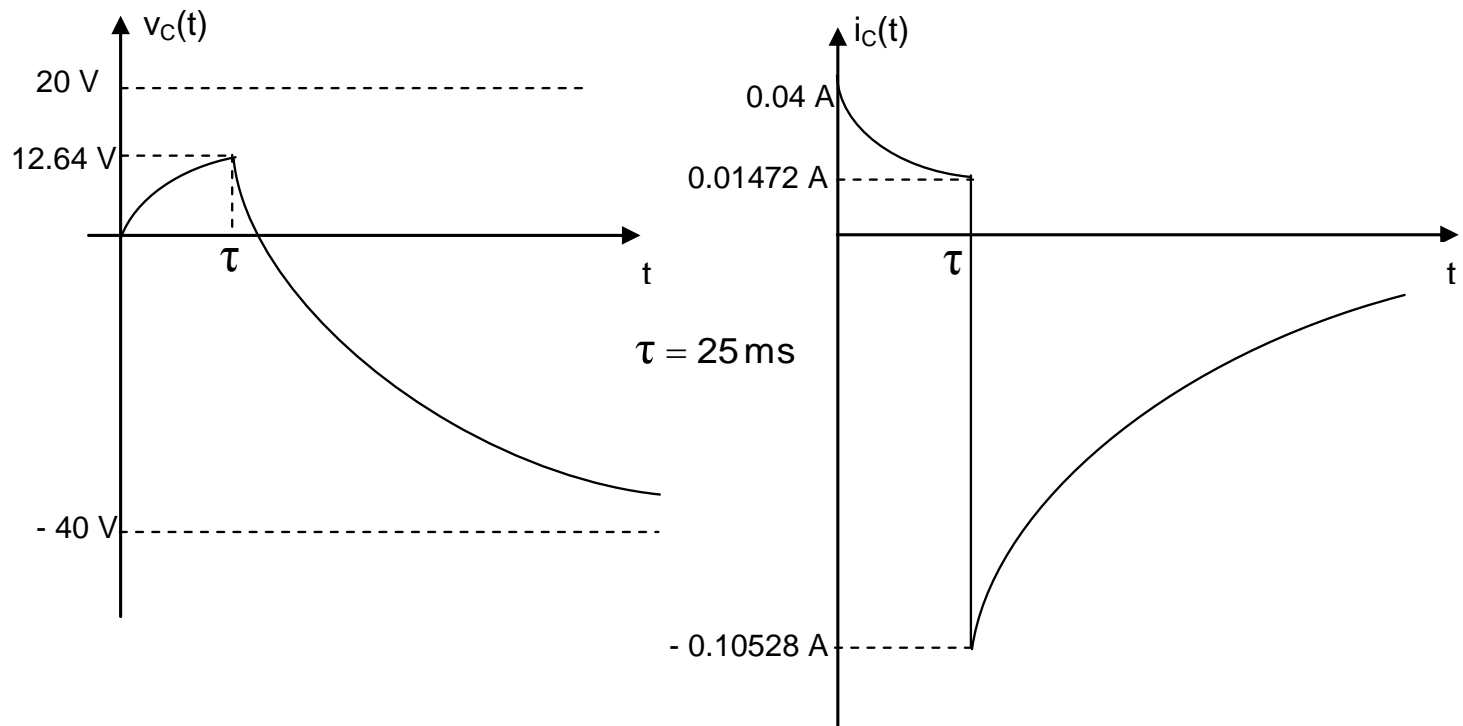
$$i_C(0.00025^-) = 0.04 e^{-1} = 0.01472 \text{ A}$$

For the second switching period,

$$v_C(t') = 52.64 e^{-4000 t'} - 40 \text{ V}$$

$$i_C(t') = 0.5 \times 10^{-6} \times (-52.64 \times 4000 e^{-4000 t'}) = -0.10528 e^{-4000 t'} \text{ A}$$

$$\text{i.e. } i_C(t - 0.00025) = -0.10528 e^{-4000 (t - 0.00025)} \text{ A} \quad i_C(0.00025^+) = -0.10528 \text{ A}$$



Note: At the switching time, voltage across the capacitor does not have discontinuity i.e. $v_C(0.25 \times 10^{-3})^- = v_C(0.25 \times 10^{-3})^+$. On the other hand, the current through the capacitor has discontinuity at the instant of switching. The current just before switching and just after switching can be calculated by considering the circuit conditions at the respective time.

At time $t = (0.25 \times 10^{-3})^-$, current $i = \frac{20 - 12.64}{500} = 0.01472 \text{ A}$

At time $t = (0.25 \times 10^{-3})^+$, current $i = \frac{-40 - 12.64}{500} = -0.10528 \text{ A}$

RC Circuit	RL Circuit
$\tau = RC \quad \alpha = 1 / RC$	$\tau = L / R \quad \alpha = R / L$
Switching at $t = 0 \quad v_C(0^+) = v_C(0^-)$	Switching at $t = 0 \quad i_L(0^+) = i_L(0^-)$
With DC, at SS capacitor acts as open circuit	With DC, at SS inductor acts as short circuit
$v_C(0) \neq 0; \quad v_C(\infty) = 0; \quad \text{Then}$ $v_C(t) = v_C(0) e^{-\frac{1}{RC} t}$	$i_L(0) \neq 0; \quad i_L(\infty) = 0; \quad \text{Then}$ $i_L(t) = i_L(0) e^{-\frac{R}{L} t}$
$v_C(0) = 0; \quad v_C(\infty) \neq 0; \quad \text{Then}$ $v_C(t) = v_C(\infty) (1 - e^{-\frac{1}{RC} t})$	$i_L(0) = 0; \quad i_L(\infty) \neq 0; \quad \text{Then}$ $i_L(t) = i_L(\infty) (1 - e^{-\frac{R}{L} t})$
$v_C(0) \neq 0; \quad v_C(\infty) \neq 0; \quad \text{Then}$ $v_C(t) = v_C(\infty) + [v_C(0) - v_C(\infty)] e^{-\frac{1}{RC} t}$	$i_L(0) \neq 0; \quad i_L(\infty) \neq 0; \quad \text{Then}$ $i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-\frac{R}{L} t}$
$i_C(t) = C \frac{dv_C(t)}{dt}$	$v_L(t) = L \frac{di_L(t)}{dt}$

7.5 TRANSIENT IN RL CIRCUIT

Now we shall consider RL circuit for the transient analysis. As stated earlier,

1. The current in an inductor cannot have discontinuity at the time when switching occurs.
2. With dc excitation, at steady state, inductor will act as a short circuit.

Now also we shall end up with first order DE whose solution will be exponential in nature.

Source free circuit

A circuit that does not contain any source is called a source free circuit. Consider the circuit shown in Fig. 7.35 (a). Let us assume that the circuit was in steady state condition with the switch is in position S_1 for a long time. Now the inductor acts as short circuit and it carries a current of $\frac{E}{R}$.

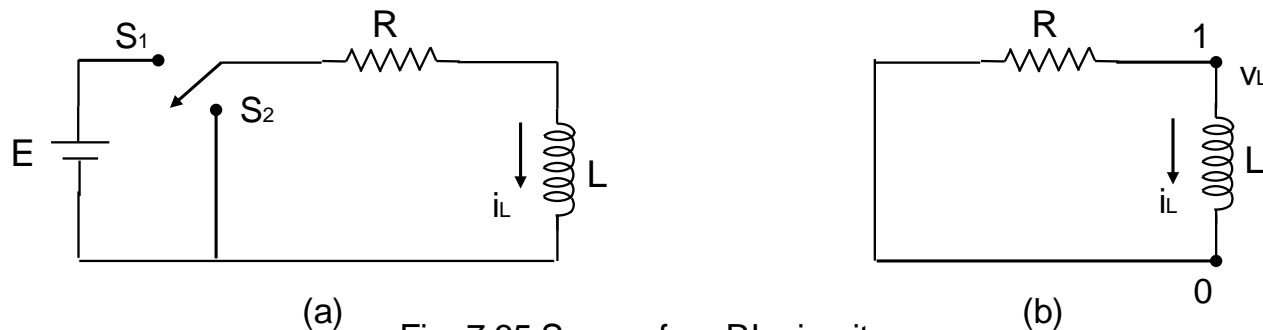
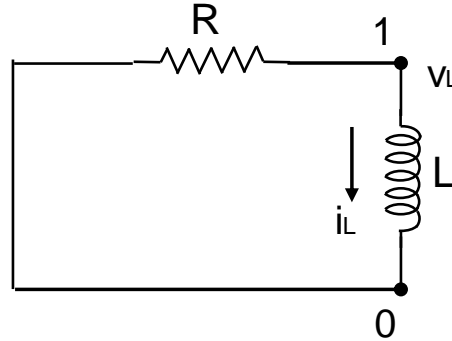


Fig. 7.35 Source free RL circuit.

Suddenly, at time $t = 0$, the switch is moved to position S_2 . The current through the inductor and the voltage across the inductor are designated as i_L and v_L respectively. The current through the inductor will be continuous. Hence



$$i_L(0^+) = i_L(0^-) = \frac{E}{R}$$

(7.49)

The circuit for time $t > 0$ is shown above. We are interested in finding the current through the inductor as a function of time. Later, if required, voltage across the inductor can be calculated from $v_L = L \frac{di}{dt}$. The mesh equation for the circuit is

$$R i_L + L \frac{di_L}{dt} = 0 \quad (7.50)$$

$$\text{i.e. } \frac{di_L}{dt} + \frac{R}{L} i_L = 0 \quad (7.51)$$

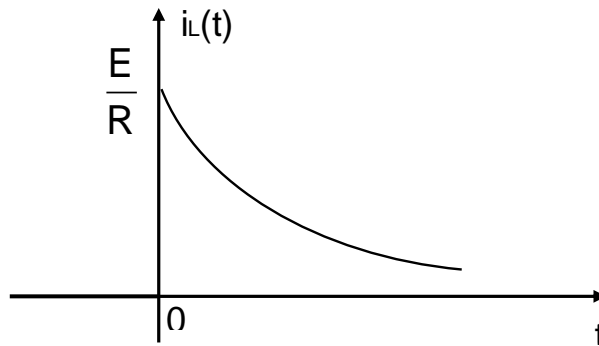
We need to solve the above equation with the initial condition

$$i_L(0^+) = \frac{E}{R} \quad (7.52)$$

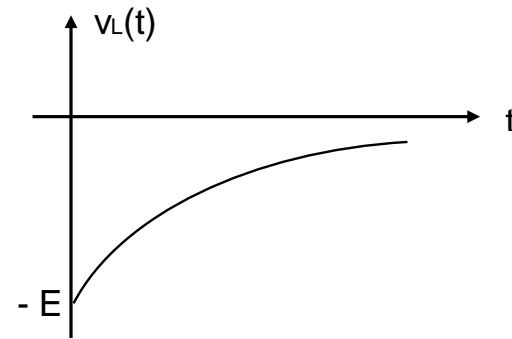
The structure of the equation (7.51) is the same as Eq. (7.18). In this case, the time constant, τ is $\frac{L}{R}$. The inductor current exponentially decays from the initial value of $\frac{E}{R}$ to the final value of zero. Thus the solution of equation 7.51 yields

$$i_L(t) = \frac{E}{R} e^{-\frac{R}{L}t} \quad (7.53)$$

The plot of inductor current is shown in Fig. (a).



(a)



(b)

It can be seen that the dimension of L / R is time. Dimensionally

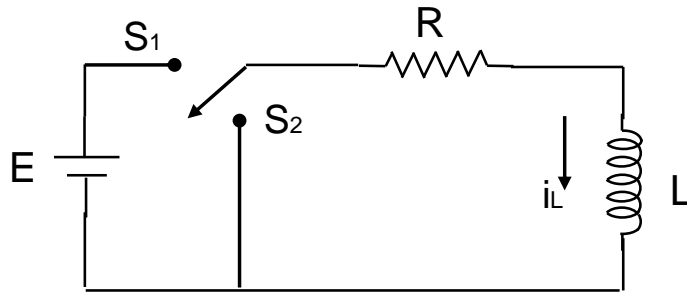
$$\frac{L}{R} = \frac{\text{Flux linkage}}{\text{amp.}} \frac{\text{amp.}}{\text{volt}} = \frac{\text{Flux linkage}}{\text{Flux linkage / sec}} = \text{sec.}$$

The voltage across the inductor is:
$$v_L(t) = L \frac{di}{dt} = L \frac{E}{R} \left(-\frac{R}{L} \right) e^{-\frac{R}{L}t} = -E e^{-\frac{R}{L}t} \quad (7.54)$$

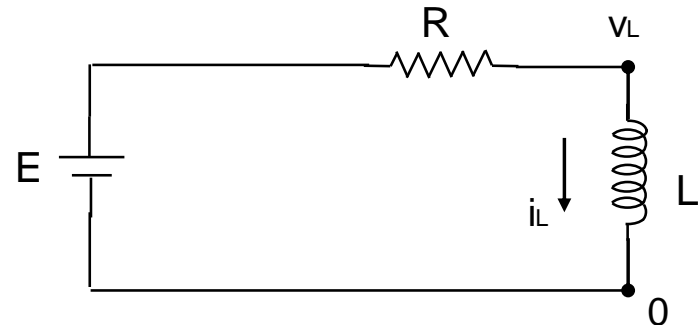
The plot of the voltage across the inductor is shown in Fig. (b).

Driven circuit

Consider the circuit shown in Fig. 7.37 (a). After the circuit has attained the steady state with the switch in position S_2 , the switch is moved to position S_1 at time $t = 0$. We like to find the inductor current for time $t > 0$.



(a)



(b)

Fig. 7.37 Driven RL circuit.

Since the current through the inductor must be continuous


$$i_L(0^+) = i_L(0^-) = 0 \quad (7.55)$$

The circuit for time $t > 0$ is shown in Fig. 7.37 (b). The mesh equation is

$$R i_L + L \frac{di_L}{dt} = E \quad (7.56)$$

$$\text{i.e. } \frac{di_L}{dt} + \frac{R}{L} i_L = \frac{E}{L} \quad (7.57)$$

We need to solve the above DE with the initial condition $i_L(0) = 0$


$$\frac{di_L}{dt} + \frac{R}{L} i_L = E$$

$$i_{cs} = K e^{-\frac{R}{L}t} \text{ and } i_{ps} = A$$

Substituting i_{ps} in the DE, we get

$$0 = -\frac{R}{L} A = \frac{E}{L} \text{ and hence } A = \frac{E}{R}$$

This gives, $i_{ps} = E / R$

$$\text{The total solution is } i_L(t) = K e^{-\frac{R}{L}t} + \frac{E}{R}$$

Using the initial condition in the above, we get

$$0 = K + \frac{E}{R} \text{ i.e. } K = -\frac{E}{R}$$

Therefore the inductor current is

$$i_L(t) = -\frac{E}{R} e^{-\frac{R}{L}t} + \frac{E}{R} = \frac{E}{R} (1 - e^{-\frac{R}{L}t}) \quad (7.58)$$

Inductor current $i_L(t)$ exponentially increases from 0 to $\frac{E}{R}$ with time constant, $\tau = \frac{L}{R}$ as shown in Fig. 7.38 (a).

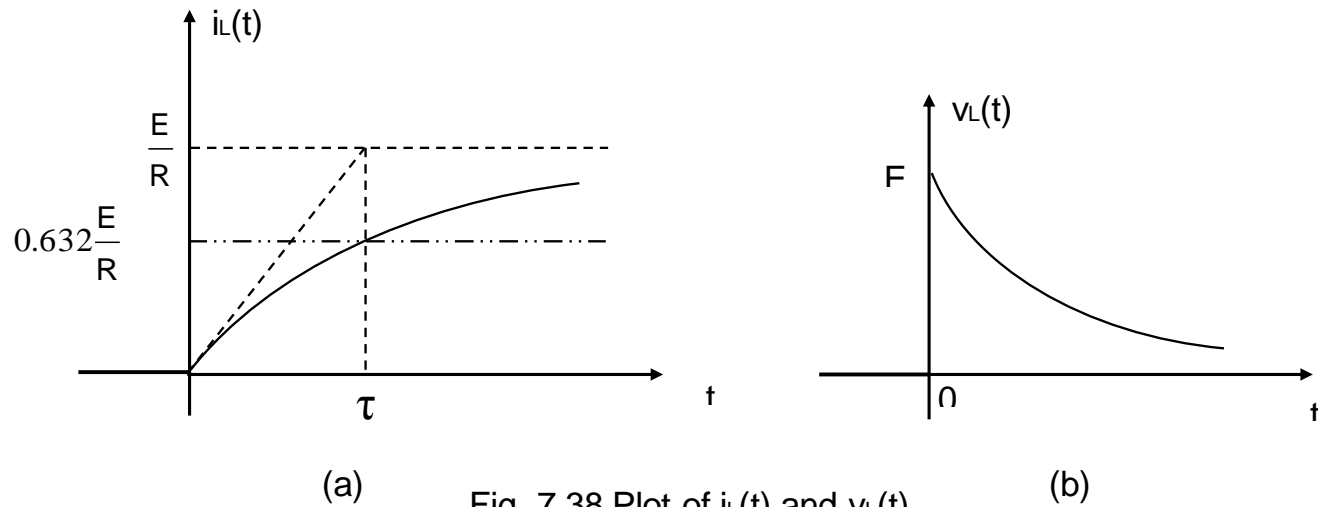


Fig. 7.38 Plot of $i_L(t)$ and $v_L(t)$.

Now, the voltage across the inductor is obtained as

$$v_L(t) = L \frac{di}{dt} = L \frac{E}{R} \frac{R}{L} e^{-\frac{R}{L}t} = E e^{-\frac{R}{L}t} \quad (7.59)$$

It can be seen that the voltage $v_L(t)$ exponentially decreases from E to zero with the time constant, $\tau = \frac{L}{R}$ as shown in Fig. 7.38 (b).

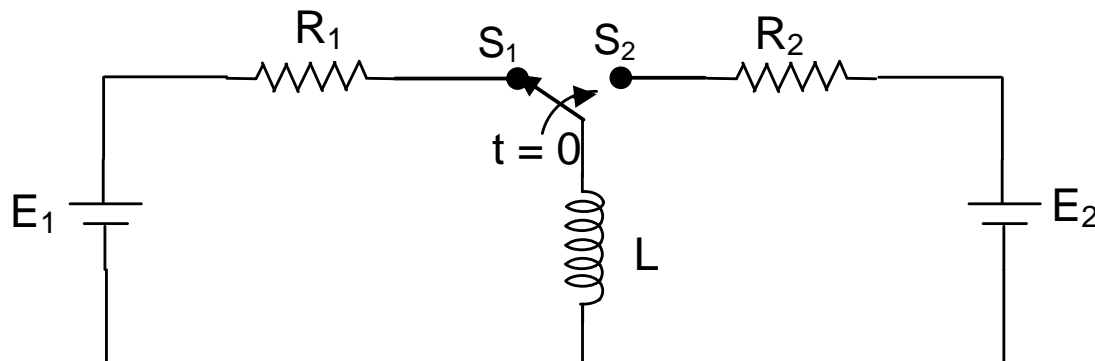
It is to be noted that the initial and the final values of the inductor current and the voltage across it can be readily computed by considering the circuit condition at that time.

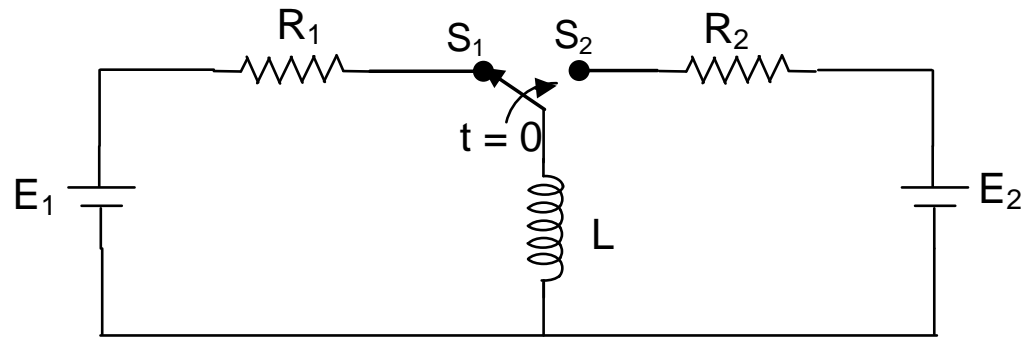
More general case of finding the inductor current

In the previous discussion, it was assumed that the initial inductor current $i_L(0) = 0$.

There may be very many situations wherein initial inductor current is not zero.

The circuit arrangements can cause non-zero initial inductor current. For this purpose consider the circuit shown below. The switch was in position S_1 for a long time. It is moved from position S_1 to S_2 at time $t = 0$.





We shall assume the following:

1. At time $t = 0^-$ the circuit was at steady state condition with the switch in position S_1
2. After switching to position S_2 , the circuit is allowed to reach the steady state condition

Thus, we are interested about the transient analysis for one switching period only.

Initial inductor current $i_L(0)$ is E_1 / R_1 and the final inductor current $i_L(\infty)$, will be E_2 / R_2 .

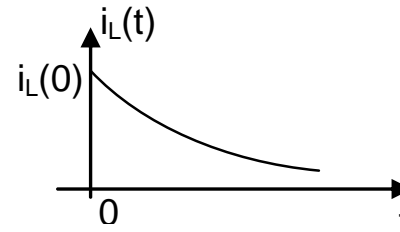
The more general expression for the inductor current can be obtained as

$$i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-\frac{R_2}{L}t} \quad (7.63)$$

Summary of formulae useful for transient analysis on RL circuits

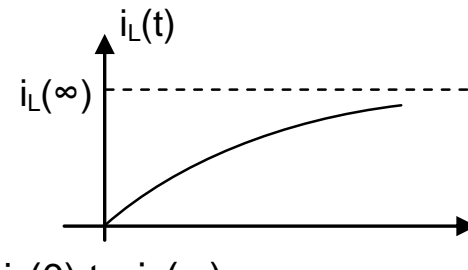
1. Time constant $\tau = L / R$ Hence $\alpha = R / L$
2. When the inductor current is decaying from the initial value of $i_L(0)$ to zero

$$i_L(t) = i_L(0) e^{-\frac{R}{L} t}$$



3. When the inductor current is exponentially increasing from zero to $i_L(\infty)$

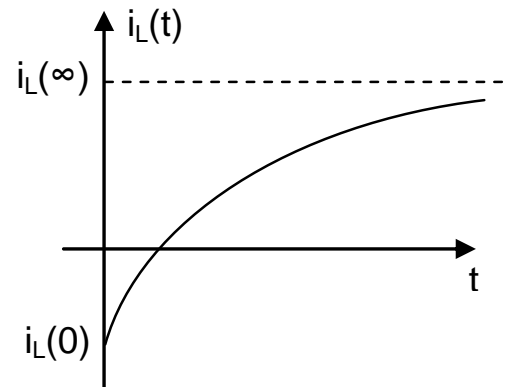
$$i_L(t) = i_L(\infty) (1 - e^{-\frac{R}{L} t})$$



4. When the inductor current changes from $i_L(0)$ to $i_L(\infty)$

$$i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-\frac{R}{L} t}$$

Plot of $i_L(t)$ depends on values of $i_L(0)$ and $i_L(\infty)$



5. Inductor voltage $v_L(t) = L \frac{di_L(t)}{dt}$

Example 7.13 An RL circuit with $R = 12 \, \Omega$ has time constant of 5 ms. Find the value of the inductance.

Solution $R = 12 \, \Omega$; Time constant, $L / R = 5 \times 10^{-3} \, \text{s}$

Inductance $L = 12 \times 5 \times 10^{-3} = 60 \, \text{mH}$

Example 7.14

In an RL circuit having time constant 400 ms the inductor current decays and its value at 500 ms is 0.8 A. Find the equation of $i_L(t)$ for $t > 0$.

Solution $L / R = 400 \times 10^{-3} \, \text{s}$; $R / L = 2.5 \, \text{s}^{-1}$; As $i_L(t)$ decays, $i_L(t) = i_L(0) e^{-\frac{R}{L}t}$

When $t = 500 \, \text{ms}$, $i_L(t) = 0.8 \, \text{A}$. Using this

$$0.8 = i_L(0) e^{-2.5 \times 0.5} = i_L(0) e^{-1.25} = 0.2865 i_L(0)$$

$$\text{Thus } i_L(0) = 0.8 / 0.2865 = 2.7923 \, \text{A}$$

$$\text{Therefore } i_L(t) = 2.7923 e^{-2.5t}$$

Example 7.15 In a RL circuit with time constant of 1.25 s, inductor current increases from the initial value of zero to the final value of 1.2 A.

- (a) Calculate the inductor current at time 0.4 s, 0.8 s and 2 s.
- (b) Find the time at which the inductor current reaches 0.3 A, 0.6 A and 0.9 A.

Solution $L / R = 1.25 \text{ s}$ $i_L(0) = 0$ $i_L(\infty) = 1.2 \text{ A}$ $\alpha = 1/1.25 = 0.8 \text{ s}^{-1}$

(a) $i_L(t) = 1.2 (1 - e^{-0.8 t}) \text{ A}$

When time $t = 0.4 \text{ s}$, $i_L = 1.2(1 - e^{-0.32}) = 0.3286 \text{ A}$

When time $t = 0.8 \text{ s}$, $i_L = 1.2(1 - e^{-0.64}) = 0.5672 \text{ A}$

When time $t = 2 \text{ s}$, $i_L = 1.2(1 - e^{-1.6}) = 0.9577 \text{ A}$

- (b) Let t_1 , t_2 and t_3 be the time at which current reaches 0.3 A, 0.6 A and 0.9 A.

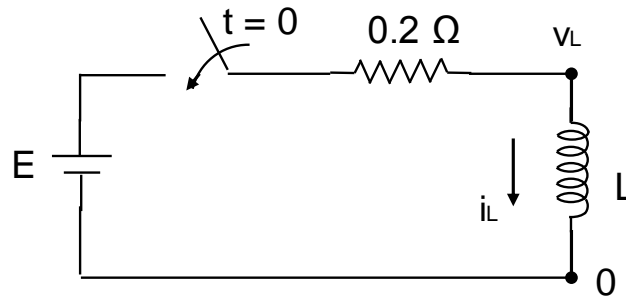
$$0.3 = 1.2 (1 - e^{-0.8 t_1}) \text{ i.e. } e^{-0.8 t_1} = 0.75 \text{ i.e. } 0.8 t_1 = 0.2877 \text{ i.e. } t_1 = 0.3596 \text{ s}$$

$$0.6 = 1.2 (1 - e^{-0.8 t_2}) \text{ i.e. } e^{-0.8 t_2} = 0.5 \text{ i.e. } 0.8 t_2 = 0.6931 \text{ i.e. } t_2 = 0.8664 \text{ s}$$

$$0.9 = 1.2 (1 - e^{-0.8 t_3}) \text{ i.e. } e^{-0.8 t_3} = 0.25 \text{ i.e. } 0.8 t_3 = 1.3863 \text{ i.e. } t_3 = 1.7329 \text{ s}$$

Example 7.16

In the RL circuit shown in Fig. below, the voltage across the inductor for $t > 0$ is given by $v_L(t) = 0.16 e^{-200t}$ V. Determine the value of the inductor L and obtain the equation for current $i_L(t)$. Also compute the value of voltage E .



Solution $v_L(t) = 0.16 e^{-200t}$ V; $R = 0.2 \Omega$ $\alpha = \frac{R}{L} = 200$; i.e. $L = \frac{0.2}{200}$ H = 1 mH

When the switch is closed inductor current exponentially increases from 0 to $i_L(\infty)$. It is

$$i_L(t) = i_L(\infty) (1 - e^{-\frac{R}{L}t}) \quad \text{Also } v_L(t) = L \frac{di_L}{dt} = L i_L(\infty) \frac{R}{L} e^{-\frac{R}{L}t} = R i_L(\infty) e^{-\frac{R}{L}t}$$

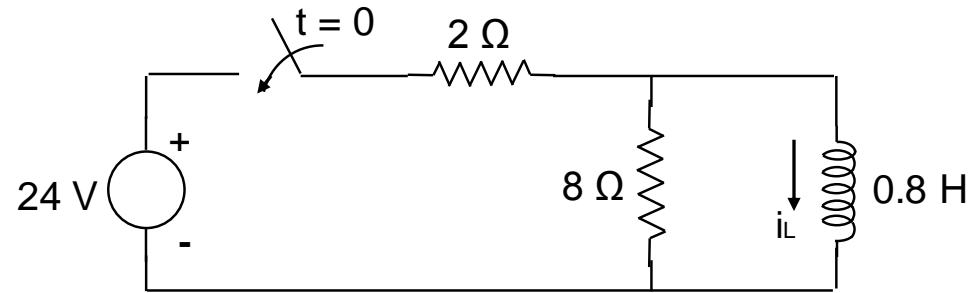
Comparing $v_L(t) = R i_L(\infty) e^{-\frac{R}{L}t}$ with $v_L(t) = 0.16 e^{-200t}$ V

Therefore, $0.2 i_L(\infty) = 0.16$ i.e. $i_L(\infty) = 0.16 / 0.2 = 0.8$ A

Thus, $i_L(t) = 0.8 (1 - e^{-200t})$

Also $i_L(\infty) = \frac{E}{0.2}$ Therefore, $\frac{E}{0.2} = 0.8$; Thus $E = 0.16$ V

Example 7.17 The switch in the circuit shown was in open position for a long time. It is closed at time $t = 0$. Find $i_L(t)$ for time $t > 0$.



Solution Current $i_L(0) = 0$

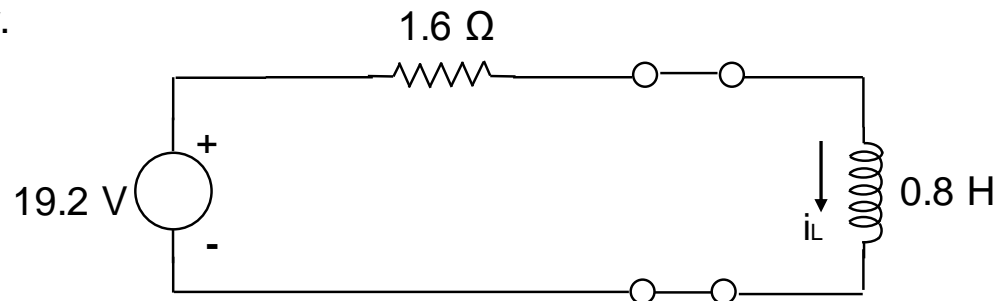
When the switch is closed, Current $i_L(\infty) = 24 / 2 = 12\text{ A}$

Thevenin's resistance $= 8 \parallel 2 = 1.6\ \Omega$ $\tau = L / R = 0.8 / 1.6 = 0.5\text{ s}$; $\alpha = 2\text{ s}^{-1}$

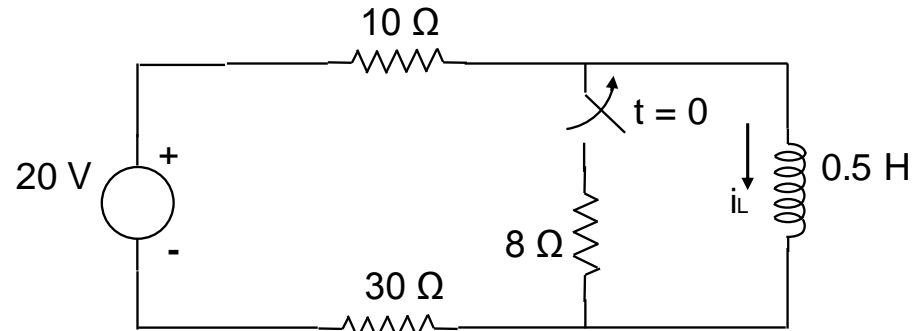
Inductor current exponentially increases from 0 to 12 A.

Current $i_L(t) = 12 (1 - e^{-2t})\text{ A}$

Same result can be obtained by getting the Thevenin's equivalent circuit for time $t > 0$ as shown in Fig. below.

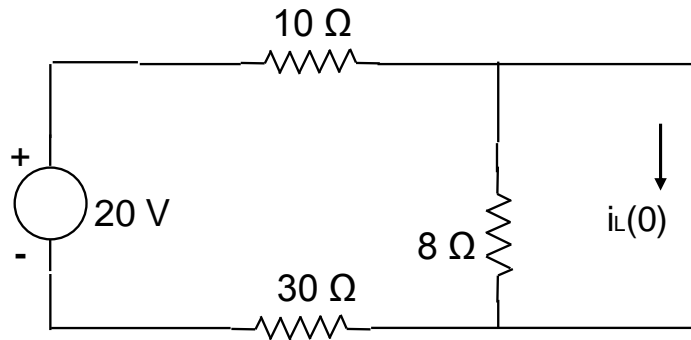


Example 7.18 The switch in the circuit shown was in closed position for a long time. Find current $i_L(t)$ for time $t > 0$.

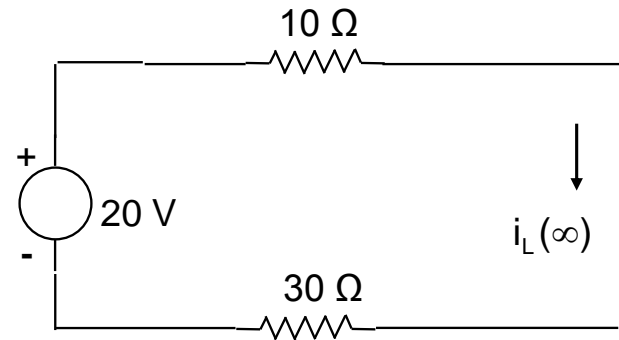


Solution

Circuit for $t = 0^-$ and $t = \infty$ are shown in Fig. (a) and (b) below.



(a)



(b)

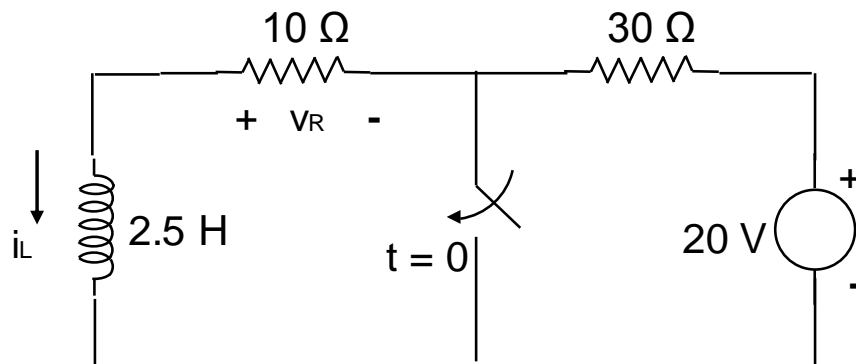
Current $i_L(0) = 20 / 40 = 0.5 \text{ A}$

Further, current $i_L(\infty) = 20 / 40 = 0.5 \text{ A}$

Therefore, current $i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-\frac{R}{L}t} = 0.5 \text{ A}$

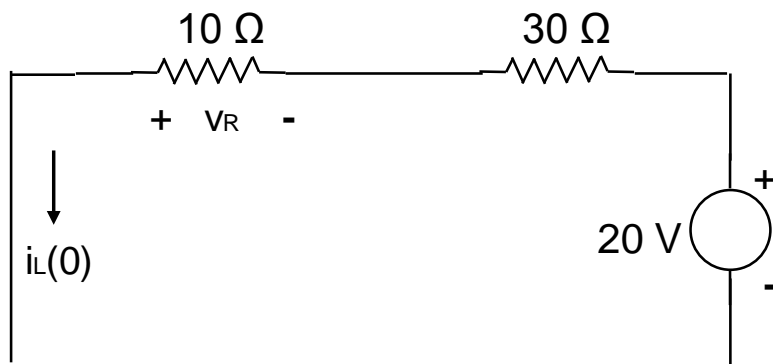
Example 7.19 In the circuit shown the switch was in open position for a long time.

Determine the current $i_L(t)$ and the voltage $v_R(t)$ for time $t > 0$.

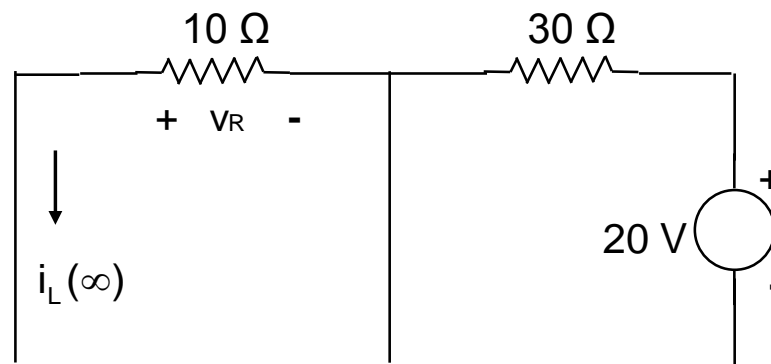


Solution

Circuit for $t = 0^-$ and $t = \infty$ are shown in Fig. (a) and (b) below.



(a)



(b)

Current $i_L(0) = 20 / (10 + 30) = 0.5$ A; Current $i_L(\infty) = 0$: Thevenin's resistance = 10Ω

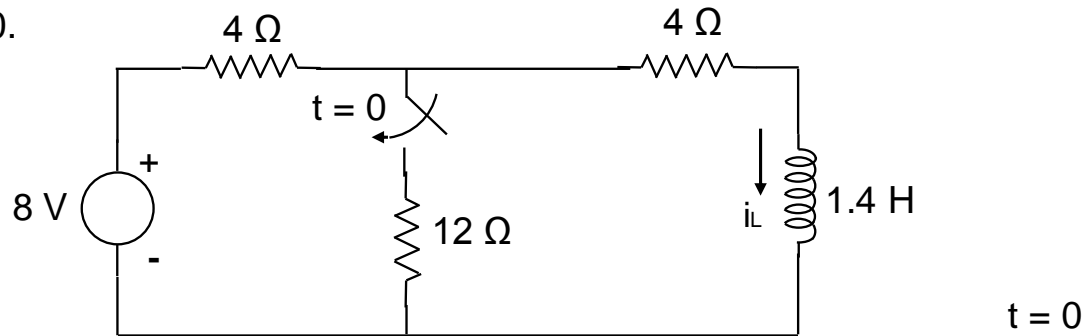
Time constant = $L / R = 2.5 / 10 = 0.25$ s; $\alpha = 4 \text{ s}^{-1}$

Thus $i_L(t) = 0.5 e^{-4t}$ A

Voltage $v_R(t) = -10 i_L(t) = -5 e^{-4t}$ V

Example 7.20

The circuit shown was in steady state condition with the switch open. Find the inductor current for time $t > 0$.



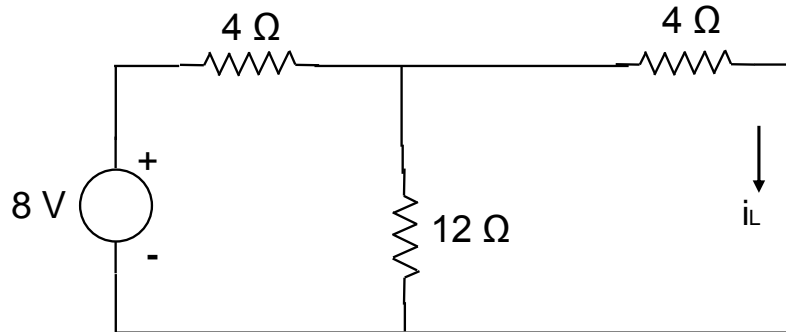
Solution

$$\text{Current } i_L(0) = 8 / (4 + 4) = 1 \text{ A}$$

Circuit for $t = \infty$ is

$$i_T = 8 / 7 = 1.1429 \text{ A}$$

$$\begin{aligned} i_L(\infty) &= (12/16) \cdot 1.1429 \text{ A} \\ &= 0.8571 \text{ A} \end{aligned}$$

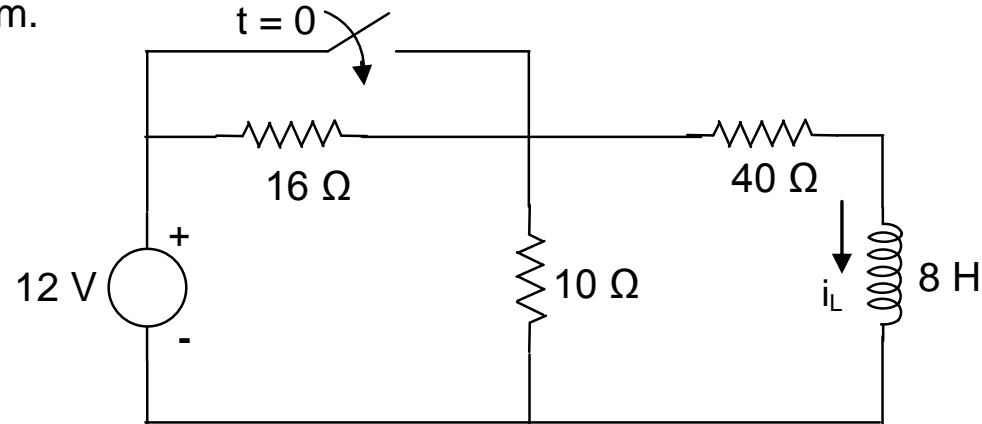


$$\text{Thevenin's resistance wrt inductor} = 4 + 3 = 7 \Omega$$

$$\text{Time constant } L / R = 1.4 / 7 = 0.2 \text{ s; } \alpha = 5 \text{ s}^{-1}$$

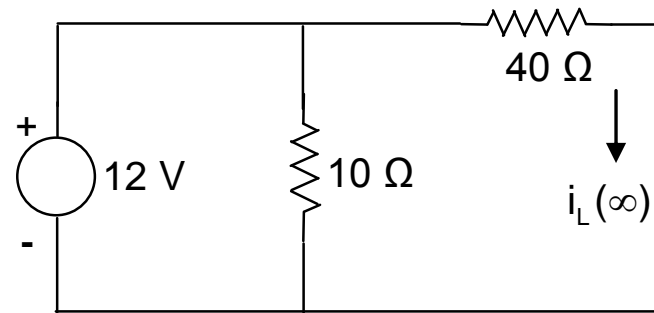
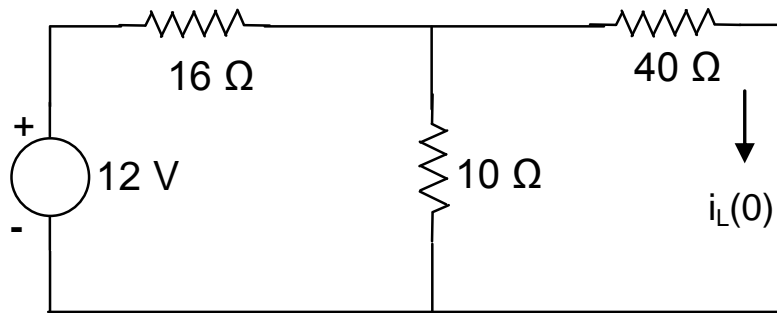
$$\begin{aligned} \text{Current } i_L(t) &= i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-\frac{R}{L}t} = 0.8571 + [1 - 0.8571] e^{-5t} \text{ A} \\ &= 0.8571 + 0.1429 e^{-5t} \text{ A} \end{aligned}$$

Example 7.21 With the switch open, the circuit shown below was in steady state condition. At time $t = 0$, the switch is closed. Find the inductor current for time $t > 0$ and sketch its wave form.



Solution

Circuit for $t = 0^-$ and $t = \infty$ are shown in Fig. (a) and (b).



To find $i_L(0)$: $R_T = 16 + 8 = 24 \Omega$; $I_T = 12 / 24 = 0.5 \text{ A}$; $i_L(0) = 0.5 \times \frac{10}{50} = 0.1 \text{ A}$

To find $i_L(\infty)$; $12 / 40 = 0.3 \text{ A}$;

Further, $R_{Th} = 40 \Omega$

Time constant = $L / R_{Th} = 8 / 40 = 0.2 \text{ s}$ $\alpha = 5 \text{ s}^{-1}$

$$\begin{aligned}\text{Current } i_L(t) &= i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-\frac{R}{L}t} = 0.3 + [0.1 - 0.3] e^{-5t} \\ &= 0.3 - 0.2 e^{-5t} \text{ A}\end{aligned}$$

Current wave form is shown in Fig. 7,51.

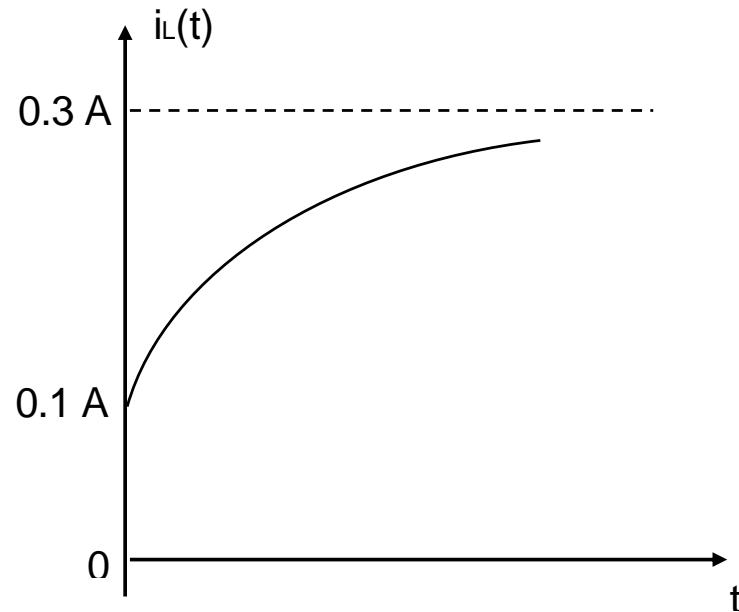
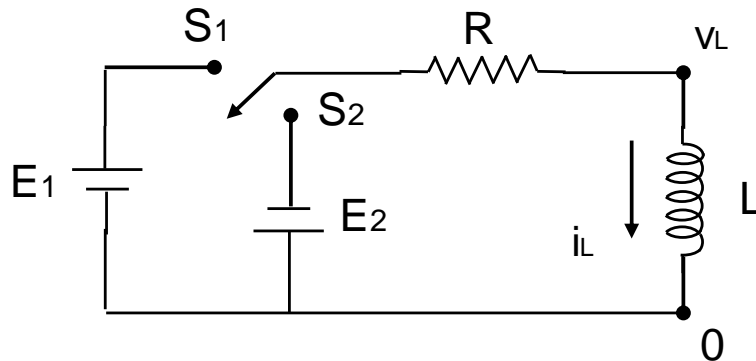


Fig. 7.51 Wave form of $i_L(t)$ - Example 7.21.

Example 7.22

For the initially relaxed circuit shown, the switch is closed on to position S_1 at time $t = 0$ and changed to position S_2 at time $t = 0.5$ ms. Obtain the equation for inductor current and voltage across the inductor in both the intervals and sketch the transients.



$$E_1 = 100 \text{ V}; E_2 = 50 \text{ V}$$

$$R = 100 \, \Omega$$

$$L = 0.2 \text{ H}$$

Solution

With the switch is in position S_1 , inductor current exponentially increases from zero to the steady state value of $100 / 100 = 1$ A. Knowing the time constant as $L / R =$

$0.2 / 100 = 1 / 500$ s, equation of inductor current in the first switching interval is

$$i_L(t) = 1 - e^{-500t} \text{ A}$$

Corresponding voltage is

$$v_L(t) = L \frac{di_L}{dt} = 0.2 \times 500 e^{-500t} \text{ V} = 100 e^{-500t} \text{ V} \quad \text{for } 0.5 \times 10^{-3} \geq t > 0$$

Therefore $i_L(0.5 \times 10^{-3}) = 1 - e^{-0.25} = 0.2212 \text{ A}$

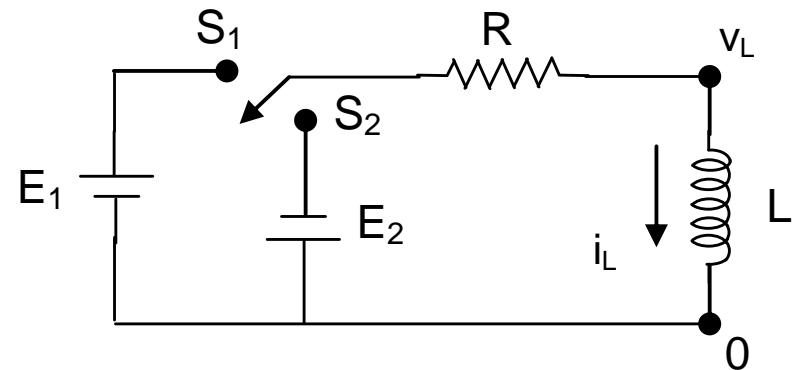
$$v_L(0.5 \times 10^{-3}) = 100 e^{-0.25} = 77.88 \text{ V}$$

Let the second switching occurs at time $t' = 0$.

Then, $t' = t - 0.5 \times 10^{-3}$

For time $t' > 0$, the mesh equation is

$$R i_L(t') + L \frac{di_L}{dt'} = -E_2 \quad \text{i.e.} \quad \frac{di_L}{dt'} + \frac{R}{L} i_L(t') = -\frac{E_2}{L} \quad \text{with } i(0) = 0.2212 \text{ A}$$



$$\frac{di_L}{dt'} + \frac{R}{L} i_L(t') = -\frac{E_2}{L} \text{ with } i(0) = 0.2212 \text{ A}$$

$$i_{cs} = K e^{-\frac{R}{L}t'} \quad \text{and} \quad i_{ps} = A$$

Substituting the particular solution to the non-homogeneous DE, we get

$$\frac{R}{L} A = -\frac{E_2}{L} \text{ i.e. } A = -\frac{E_2}{R} = -0.5$$

Complete solution is

$$i_L(t') = K e^{-500 t'} - 0.5$$

Using the initial condition

$$K - 0.5 = 0.2212 \text{ i.e. } K = 0.7212. \text{ Thus}$$

$$i_L(t') = 0.7212 e^{-500 t'} - 0.5 \text{ A}$$

$$v_L(t') = 0.2 \times (-0.7212 \times 500) e^{-500 t'} = -72.12 e^{-500 t'} \text{ V}$$

When $t' = 0$, inductor voltage = - 72.12 V

The current and voltage transients are shown in Fig. 7.53.

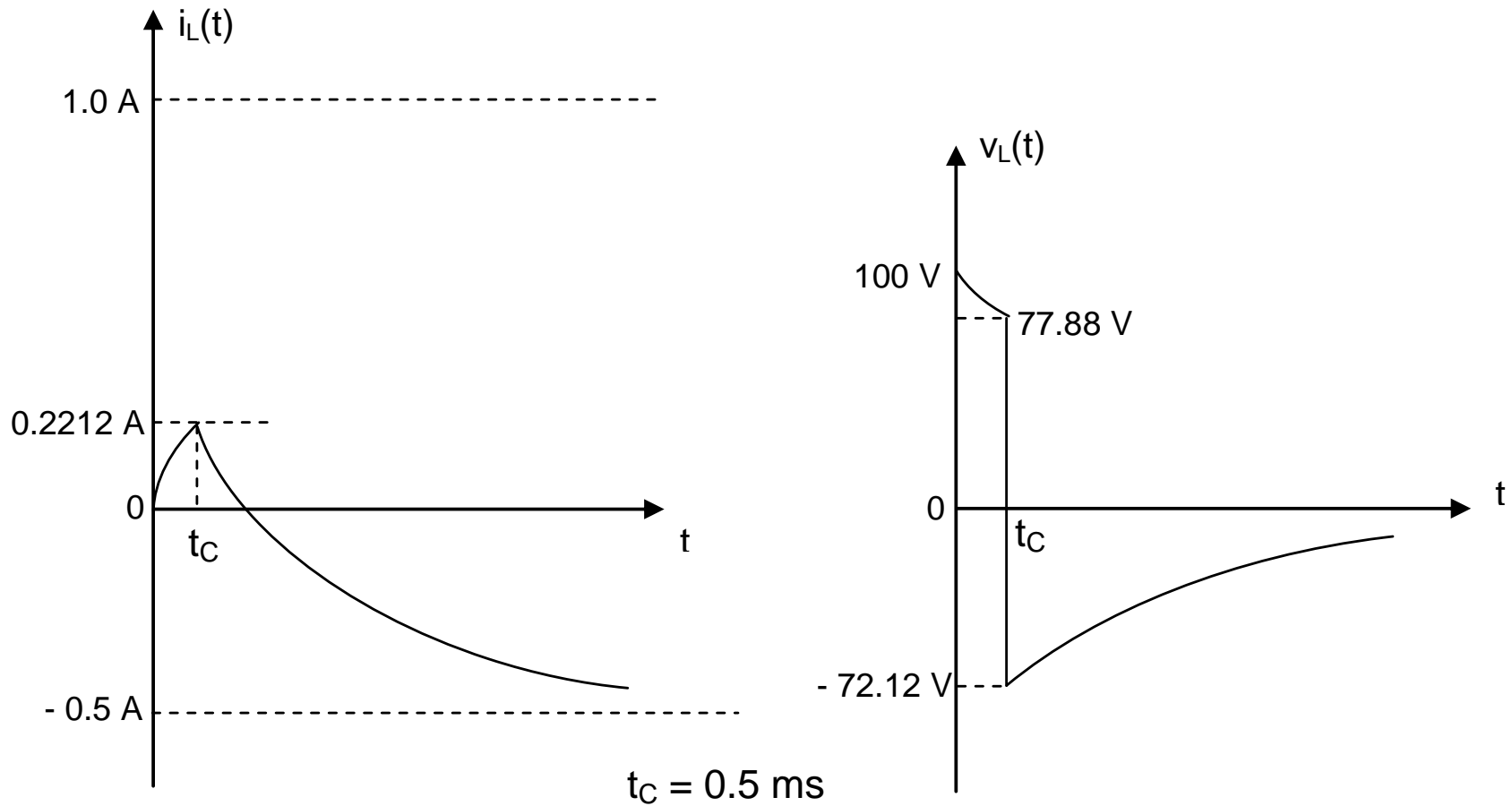


Fig. 7.53 Wave forms - Example 7.22.

7.6 LAPLACE TRANSFORM

In circuits with several capacitances and inductors, we often come across with integro-differential equations. Such equations can be rewritten as higher order DEs. The classical method of solving the DEs is rather involved. Here, the complimentary solution and the particular solution have to be determined and finally the arbitrary constants have to be obtained from the initial conditions. The Laplace Transform (LT) method is much superior to the classical method due to the following reasons.

1. Laplace transformation transforms exponential and trigonometric functions into algebraic functions.
2. Laplace transformation transforms differentiation and integration into multiplication and division respectively.
3. It transforms integro-differential equations into algebraic equations which are much simpler to handle.
4. The arbitrary constants need not be determined separately. Complete solution will be obtained directly.

The LT of $f(t)$ is defined by

$$F(s) = \int_0^{\infty} f(t) e^{-st} dt \quad (7.65)$$

The following Table 7.1 gives the LT of some important functions used quite often in transient analysis.

Table 7.1 Laplace transform of certain time functions.

Time function f(t)	Laplace transform F(s)	Time function f(t)	Laplace transform F(s)
$u(t)$	$\frac{1}{s}$	E	$\frac{E}{s}$
e^{-at}	$\frac{1}{s+a}$	e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\sin (\omega t + \theta)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\cos (\omega t + \theta)$	$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$
$\frac{df}{dt}$	$s F(s) - f(0^+)$	$\frac{d^2f}{dt^2}$	$s^2 F(s) - s f(0^+) - f'(0^+)$
$\int_0^\infty f(t) dt$	$\frac{F(s)}{s}$	$e^{-\alpha t} f(t)$	$F(s + \alpha)$
$f(t - t_1)$	$e^{-t_1 s} F(s)$	t	$\frac{1}{s^2}$

While finding inverse Laplace Transform, in many cases, as a first step, $F(s)$ is to be split into sum of functions in s . This is done using partial fraction method. The results of two cases that are used quite often are furnished below.

$$1. F(s) = \frac{s^2 + ps + q}{(s + a)(s + b)(s + c)} = \frac{K_1}{s + a} + \frac{K_2}{s + b} + \frac{K_3}{s + c} \quad (7.66)$$

$$\text{Here } K_1 = (s + a) F(s) \Big|_{s = -a}$$

$$K_2 = (s + b) F(s) \Big|_{s = -b}$$

$$K_3 = (s + c) F(s) \Big|_{s = -c} \quad (7.67)$$

$$2. F(s) = \frac{A}{s(s + B)} = \frac{k_1}{s} + \frac{k_2}{s + B} = \frac{A}{B} \frac{1}{s} - \frac{A}{B} \frac{1}{s + B} = \frac{A}{B} \left(\frac{1}{s} - \frac{1}{s + B} \right)$$

7.7 TRANSFORM IMPEDANCE AND TRANSFORM CIRCUIT

When LT method is used for transient analysis, **Transform Circuit shall be arrived first**. In the transform circuit, **all the currents and voltages are the transformed quantities of the currents and voltages**. Further, all the element parameters are replaced by their Transform Impedances. Transform impedances of the individual element shall be arrived at as discussed below.

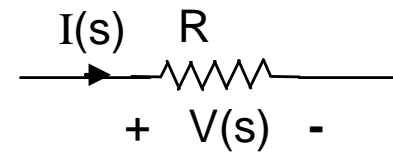
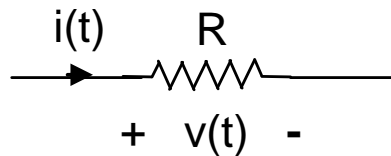
Resistor

The terminal relationship for the resistor, in time domain is

$$v(t) = R i(t) \quad (7.68)$$

$$\text{Taking LT on both sides,} \quad V(s) = R I(s) \quad (7.69)$$

Fig. below shows the terminal relationships of resistor in time and transform domains.



Inductor For an inductor, v-i relationships in time domain are

$$v(t) = L \frac{di}{dt} \quad (7.70)$$

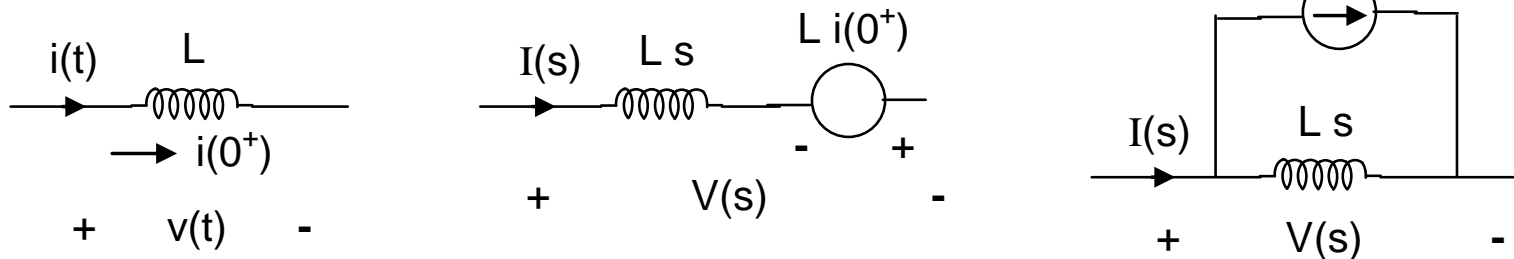
$$i(t) = \frac{1}{L} \int_0^t v \, dt + i(0^+) \quad (7.71)$$

where $i(0^+)$ is the current flowing through the inductor at time $t = 0^+$. On taking LT of these equations, we get

$$V(s) = L s I(s) - L i(0^+) \quad (7.72)$$

$$I(s) = \frac{V(s)}{L s} + \frac{i(0^+)}{s} \quad (7.73)$$

Note that above two equations are not different. Fig. below shows the representation of the terminal relationship of inductor in time and transform domains.



It is to be noted that both the transform domain circuits shown above are equivalent of each other. One can be obtained from the other using source transformation.

Capacitor For a capacitor, v-i relationships in time domain are

$$i(t) = C \frac{dv}{dt} \quad (7.74)$$

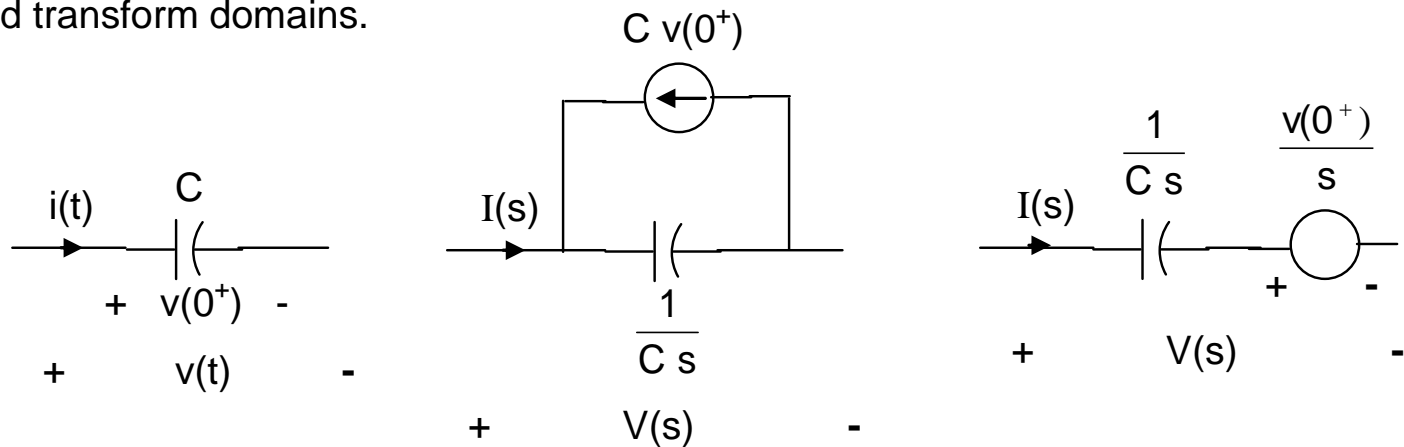
$$v(t) = \frac{1}{C} \int_0^t i \, dt + v(0^+) \quad (7.75)$$

where $v(0^+)$ is the voltage across the capacitor at time $t = 0^+$. On taking LT of these equations, we get

$$I(s) = C s V(s) - C v(0^+) \quad (7.76)$$

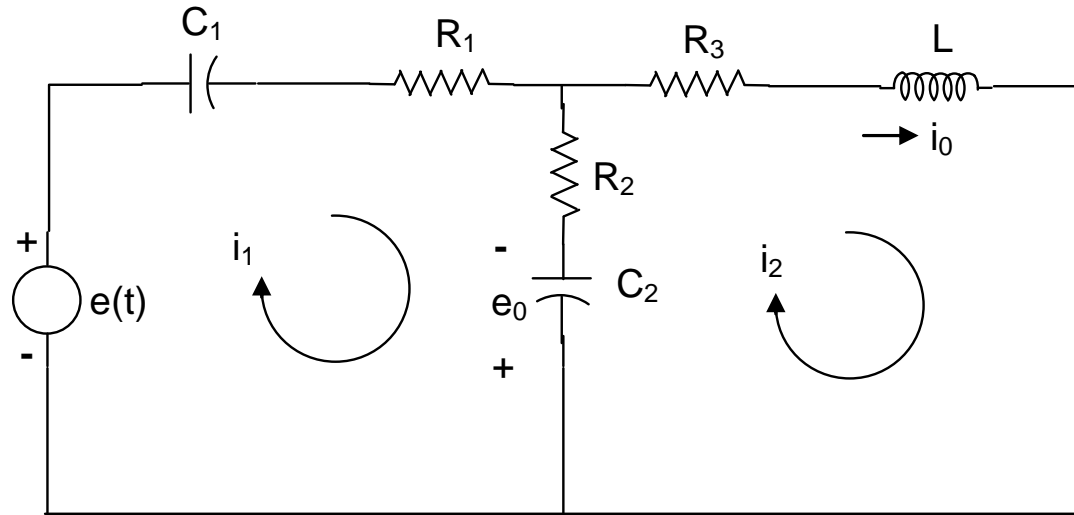
$$V(s) = \frac{I(s)}{C s} + \frac{v(0^+)}{s} \quad (7.77)$$

Note that the above two equations are not different. They are written in different form. Fig. below shows the representation of the terminal relationship of capacitor in the time and transform domains.

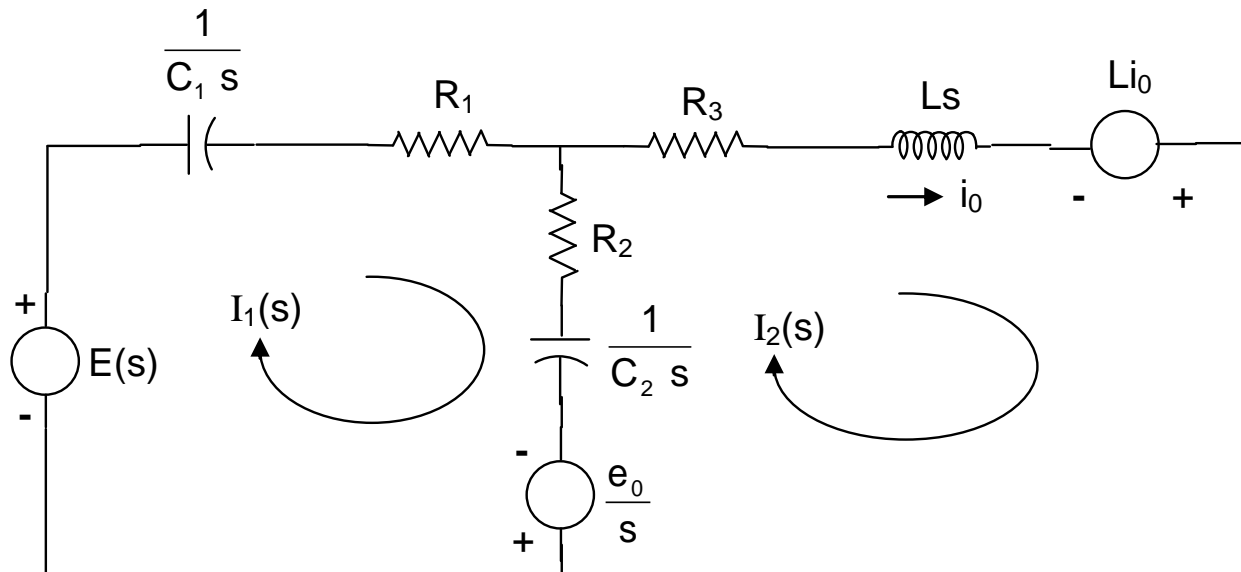


Here again, both the transform domain circuits shown are equivalent of each other. One can be obtained from the other using source transformation.

Example 7.23 For the circuit shown below, obtain the transform circuit.

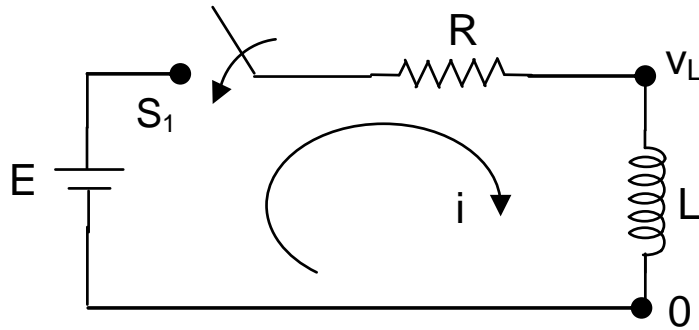


Solution Fig. below shows the transform circuit.

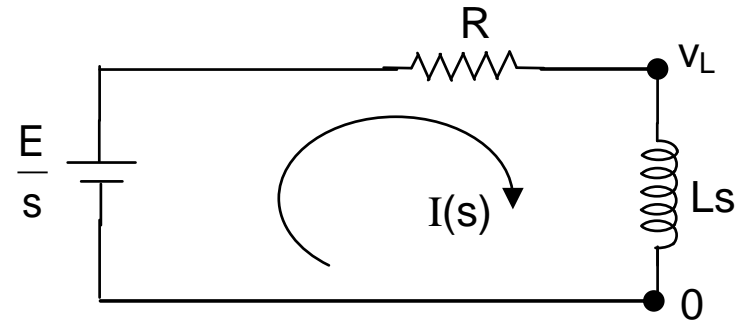


7.8.1 RL CIRCUIT

Consider the RL circuit shown in Fig. 7.59(a). Assume that the switch is closed at time $t = 0$ and assume that the current i at the time of switching is zero.



(a)



(b)

Fig. 7.59 Time domain and s domain - R-L circuit.

The transform circuit in s domain is shown in Fig. 7.59 (b). From this,

$$I(s) = \frac{E/s}{R + Ls} = \frac{E/L}{s(s + \frac{R}{L})} = \frac{E/L}{R/L} \left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}} \right) = \frac{E}{R} \left(\frac{1}{s} - \frac{1}{s + \frac{R}{L}} \right) \quad (7.78)$$

Taking inverse LT

$$i(t) = \frac{E}{R} (1 - e^{-\frac{R}{L}t}) \quad (7.79)$$

Thus, inductor current rises exponentially with time constant L / R .

Voltage across the inductor is given by

$$V(s) = L s I(s) = \frac{E}{s + \frac{R}{L}} \quad (7.80)$$

Taking inverse LT $v_L(t) = E e^{-\frac{R}{L}t}$ (7.81)

Inductor voltage increases exponentially with time constant L / R . The current and voltage transients are shown in Fig. 7.60.

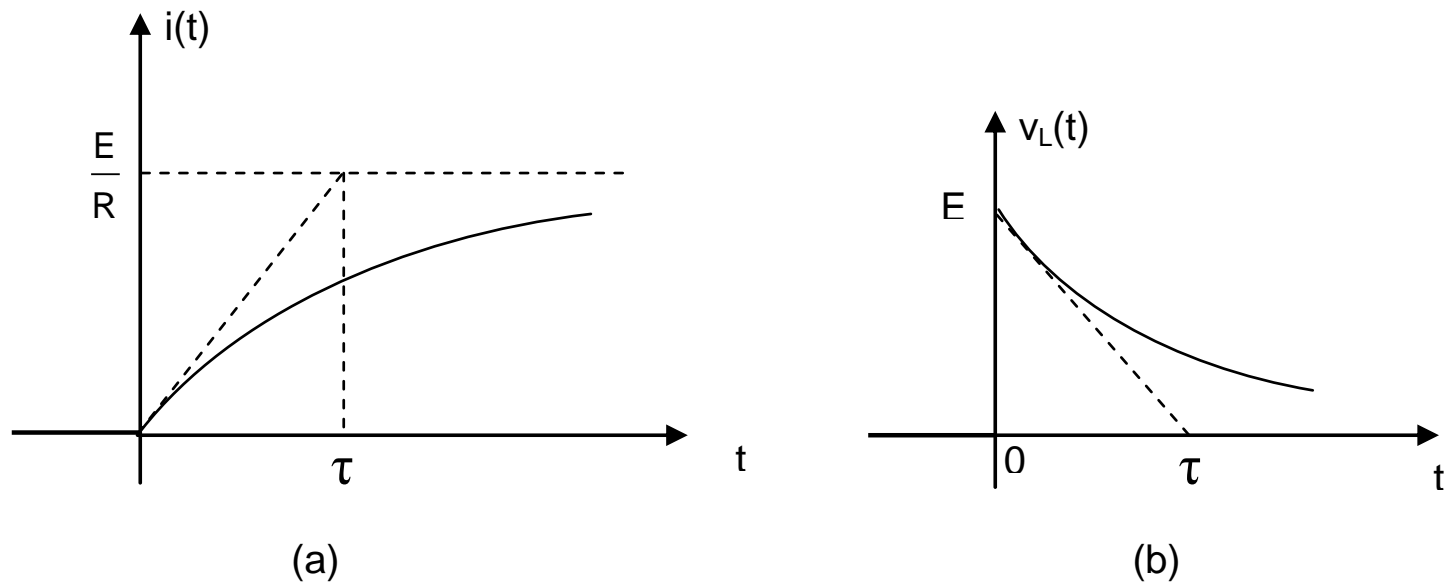
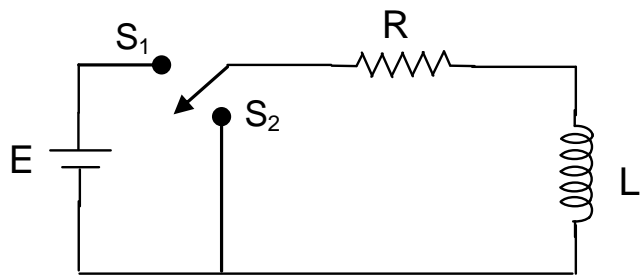


Fig. 7.60 Plot of $i_L(t)$ and $v_L(t)$.

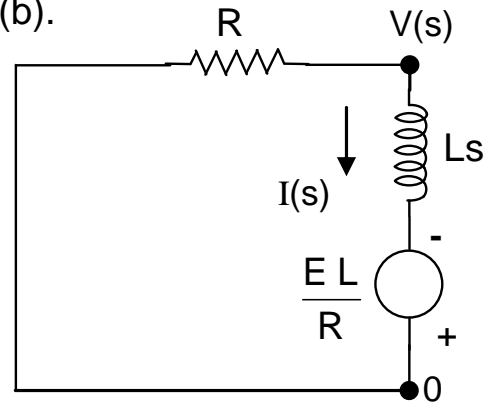
Consider the circuit shown in Fig.(a). Let us say that with the switch in position S_1 , steady state condition is reached. The current flowing through the inductor is E / R . At time $t = 0$, the switch is turned to position S_2 . Then

$$i(0^+) = i(0^-) = E / R$$

The transform circuit for time $t > 0$ is shown in Fig. (b).



(a)



(b)

Considering the transformed circuit

$$I(s) = \frac{\frac{EL}{R}}{R + Ls} = \frac{\frac{E}{R}}{s + \frac{R}{L}} \quad (7.82)$$

Taking inverse LT

$$i(t) = \frac{E}{R} e^{-\frac{R}{L}t} \quad (7.83)$$

The current decays exponentially with time constant L / R .

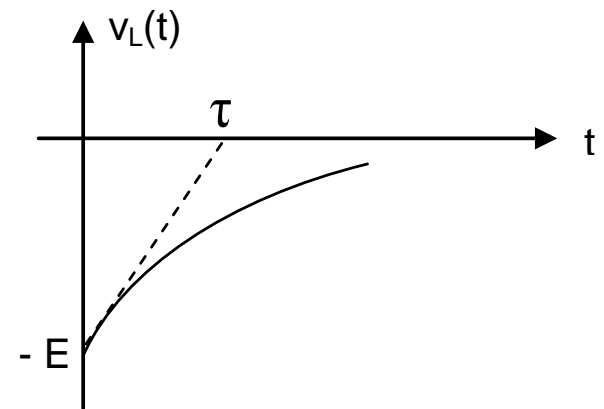
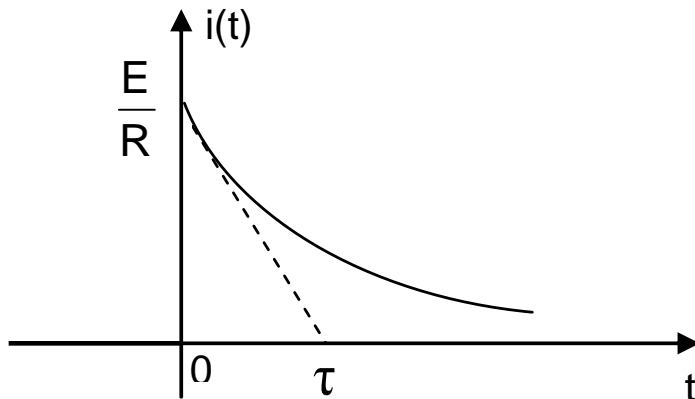
Since $R I(s) + V(s) = 0$ the voltage across the inductor is

$$V(s) = -R I(s) = - \frac{E}{s + \frac{R}{L}} \quad (7.84)$$

Taking inverse LT $v_L(t) = -E e^{-\frac{R}{L}t}$ (7.85)

The inductor voltage exponentially changes from $-E$ to zero with time constant L / R .

The current and voltage transients are given by the above two equations are shown.



Example 7.24 Initially relaxed series RL circuit with $R = 100 \, \Omega$ and $L = 20 \, \text{H}$ has dc voltage of $200 \, \text{V}$ applied at time $t = 0$. Find (a) the equation for current and voltages across different elements (b) the current at time $t = 0.5 \, \text{s}$ and $1.0 \, \text{s}$ (c) the time at which the voltages across the resistor and inductor are equal.

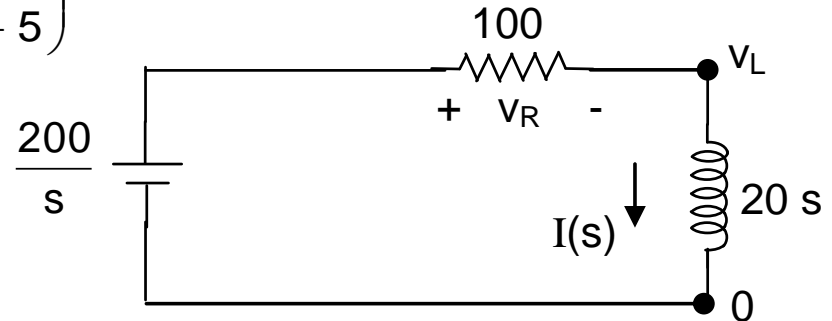
Solution Transform circuit for time $t > 0$ is shown.

$$(a) \, I(s) = \frac{\frac{200}{s}}{100 + 20s} = \frac{10}{s(s + 5)} = 2 \left(\frac{1}{s} - \frac{1}{s + 5} \right)$$

Therefore, current $i(t) = 2 (1 - e^{-5t}) \, \text{A}$

Voltage $v_R(t) = R i(t) = 200 (1 - e^{-5t}) \, \text{V}$

Voltage $v_L(t) = L \frac{di}{dt} = 20 \times 2 \times 5 e^{-5t} = 200 e^{-5t} \, \text{V}$

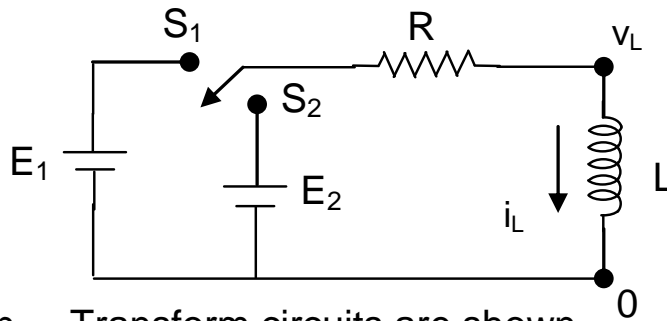


(b) $i(0.5) = 2 (1 - e^{-2.5}) = 1.8358 \, \text{A}$ $i(1.0) = 2 (1 - e^{-5}) = 1.9865 \, \text{A}$

(c) Let t_1 be the time at which $v_R(t) = v_L(t)$. Then

$200 (1 - e^{-5t_1}) = 200 e^{-5t_1}$ i.e. $e^{-5t_1} = 0.5$ This gives $t_1 = 0.1386 \, \text{s}$

Example 7.25 For the circuit shown, with zero inductor current the switch is closed on to position S_1 at time $t = 0$. At one mille second it is moved to position S_2 Obtain the equation for the currents in both the intervals.

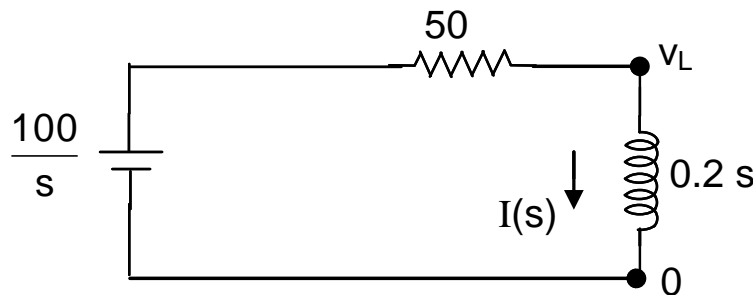


$$E_1 = 100 \text{ V}; E_2 = 50 \text{ V}$$

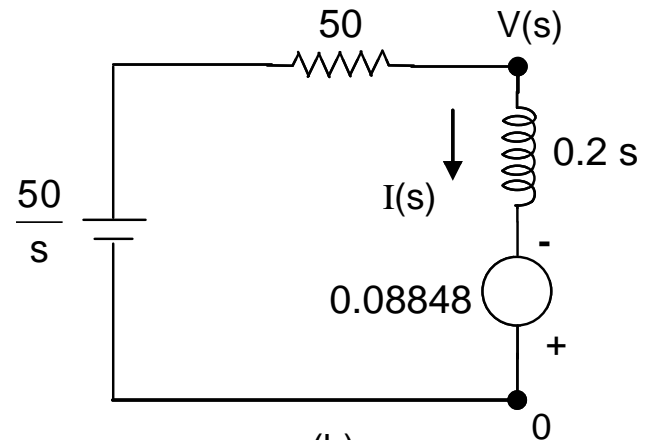
$$R = 50 \Omega$$

$$L = 0.2 \text{ H}$$

Solution Transform circuits are shown.



(a)



(b)

The transform circuit for the first interval is shown in Fig. 7.65 (a). From this

$$I(s) = \frac{\frac{100}{s}}{50 + 0.2s} = \frac{500}{s(s + 250)} = 2 \left(\frac{1}{s} - \frac{1}{s + 250} \right)$$

$$\text{Thus, } i(t) = 2 (1 - e^{-250t}) \text{ A} \quad i(0.001) = 2 (1 - e^{-0.25}) = 0.4424 \text{ A}$$

At time $t = 0.001$ s, the switch is moved to position S_2 . We shall say that this is done at time $t' = 0$. Thus $t' = 0$ implies that $t = 0$ and hence $t' = t - 0.001$.

The transform circuit for time $t' > 0$ is shown in Fig. 7.65 (b) in which

$$L i(0^+) = 0.2 \times 0.4424 = 0.08848$$

$$\text{Now, } I(s) = \frac{\frac{50}{s} + 0.08848}{50 + 0.2s} = \frac{50 + 0.08848s}{s(50 + 0.2s)} = \frac{250 + 0.4424s}{s(s + 250)} = \frac{K_1}{s} + \frac{K_2}{s + 250}$$

$$K_1 = \left. \frac{250 + 0.4424s}{s + 250} \right|_{s=0} = 1 \quad K_2 = \left. \frac{250 + 0.4424s}{s} \right|_{s=-250} = -0.5576$$

$$\text{Thus, } I(s) = \frac{1}{s} - \frac{0.5576}{s + 250}$$

Taking inverse LT we get, current $i(t') = 1 - 0.5576 e^{-250t'}$

Thus for the two intervals currents are given by

$$i(t) = 2 (1 - e^{-250t}) \text{ A} \quad 0.001 \geq t > 0$$

$$i(t) = 1 - 0.5576 e^{-250(t - 0.001)} \text{ A} \quad t > 0.001$$

Example 7.26 In the previous example, compute the voltage across the inductor in both the intervals and sketch the wave form.

Solution In the first interval, $i(t) = 2 (1 - e^{-250t})$ A

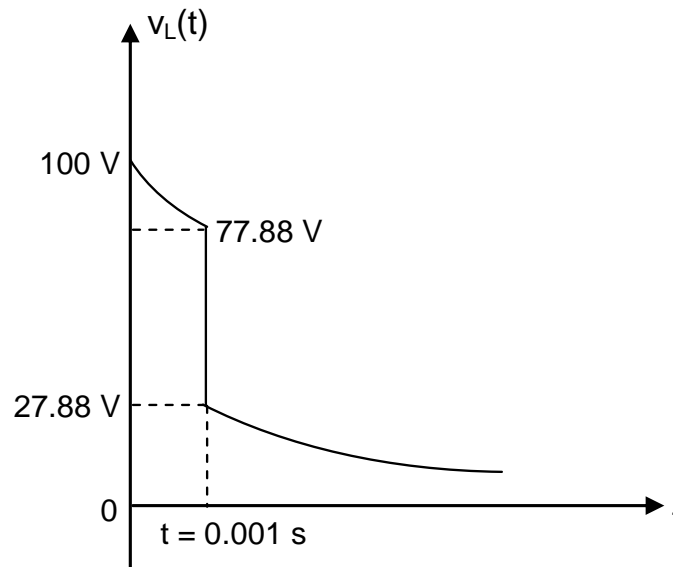
$$v_L(t) = L \frac{di}{dt} = 0.2 \times 2 \times 250 e^{-250t} = 100 e^{-250t} \text{ V} \quad v_L(0.001) = 100 e^{-0.25} = 77.88 \text{ V}$$

In the second interval, $i(t') = 1 - 0.5576 e^{-250t'}$

$$v_L(t') = L \frac{di}{dt'} = 0.2 \times 0.5576 \times 250 e^{-250t'} = 27.88 e^{-250t'} = 27.88 e^{-250(t - 0.001)} \text{ V}$$

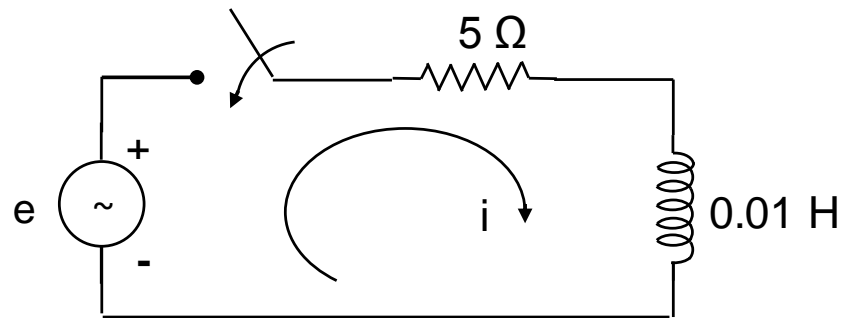
$$v_L(0.001) = v_L(t') \Big|_{t'=0} = 27.88 \text{ V}$$

The wave form of the voltage across the inductor is shown below.



Example 7.27

In the initially relaxed RL circuit shown, the sinusoidal source of $e = 100 \sin (500 t)$ V is applied at time $t = 0$. Determine the resulting transient current for time $t > 0$.



Solution

$e = 100 \sin (500 t)$ V; Its LT is

$$E(s) = \frac{100 \times 500}{s^2 + 250000} = \frac{5 \times 10^4}{s^2 + 25 \times 10^4}$$

$$\text{Impedance} = 5 + j 0.01 s$$

$$\begin{aligned}\text{Current } I(s) &= \frac{5 \times 10^4}{(s^2 + 25 \times 10^4)(5 + 0.01s)} = \frac{5 \times 10^6}{(s^2 + 25 \times 10^4)(s + 500)} \\ &= \frac{K_1 s + K_2}{s^2 + 25 \times 10^4} + \frac{K_3}{s + 500}\end{aligned}$$

$$K_3 = \left. \frac{5 \times 10^6}{s^2 + 25 \times 10^4} \right|_{s = -500} = 10$$

$$\text{Since } \frac{5 \times 10^6}{(s^2 + 25 \times 10^4)(s + 500)} = \frac{K_1 s + K_2}{s^2 + 25 \times 10^4} + \frac{10}{s + 500}$$

$$\begin{aligned}5 \times 10^6 &= (K_1 s + K_2)(s + 500) + 10(s^2 + 25 \times 10^4) \\ &= (K_1 + 10)s^2 + (500 K_1 + K_2)s + (500 K_2 + 25 \times 10^5)\end{aligned}$$

Comparing the coefficients, in LHS and RHS

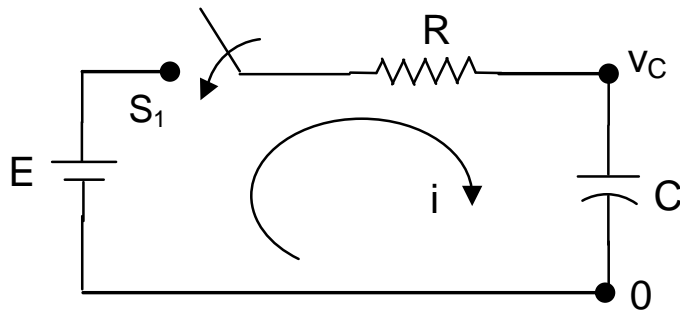
$$K_1 + 10 = 0 \text{ i.e. } K_1 = -10$$

$$500 K_1 + K_2 = 0 \text{ i.e. } K_2 = -500 K_1. \text{ Thus } K_2 = 5000$$

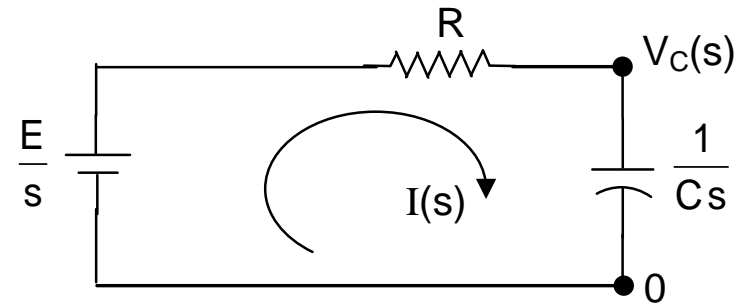
$$\text{Therefore, } I(s) = \left[\frac{-10s}{s^2 + 25 \times 10^4} + \frac{5000}{s^2 + 25 \times 10^4} + \frac{10}{s + 500} \right]$$

$$\begin{aligned}\text{On taking inverse LT, we get } i(t) &= 10 [-\cos 500 t + \sin 500 t + e^{-500t}] \text{ A} \\ &= 14.14 \sin (500 t - 45^\circ) + 10 e^{-500t} \text{ A}\end{aligned}$$

7.8.2 RC CIRCUIT Consider the RC circuit shown in Fig. 7.68 (a). Assume that the switch is closed at time $t = 0$ and assume that the voltage across the capacitor at the time of switching is zero.



(a)



(b)

Fig. 7.68 Time domain and s domain - RC circuit.

The transform circuit for time $t > 0$ is shown in Fig. 7.68 (b). From this

$$I(s) = \frac{E/s}{R + \frac{1}{Cs}} = \frac{EC}{RCs + 1} = \frac{E/R}{s + \frac{1}{RC}} \quad (7.85)$$

Taking inverse LT $i(t) = \frac{E}{R} e^{-\frac{1}{RC}t}$ (7.86)

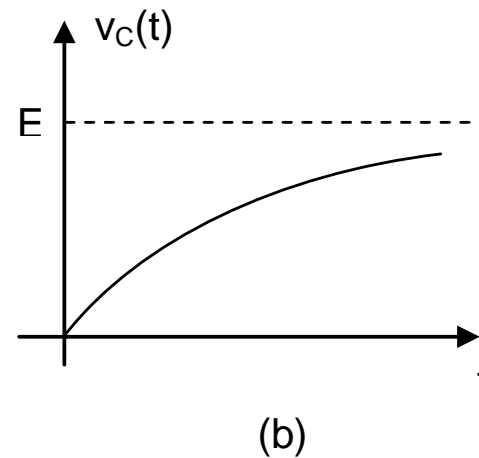
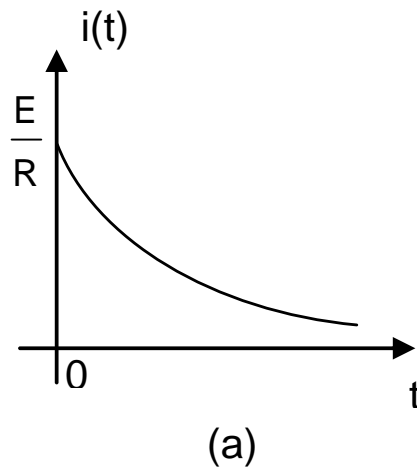
Voltage across the capacitor is $V_C(s) = \frac{1}{Cs} I(s) = \frac{E/RC}{s(s + \frac{1}{RC})} = E \left(\frac{1}{s} - \frac{1}{s + \frac{1}{RC}} \right)$ (7.87)

Voltage across the capacitor is $V_C(s) = \frac{1}{Cs} I(s) = \frac{E/RC}{s(s + \frac{1}{RC})} = E \left(\frac{1}{s} - \frac{1}{s + \frac{1}{RC}} \right)$ (7.87)

Taking inverse LT, we get the capacitor voltage as

$$v_C(t) = E (1 - e^{-\frac{1}{RC}t})$$
 (7.88)

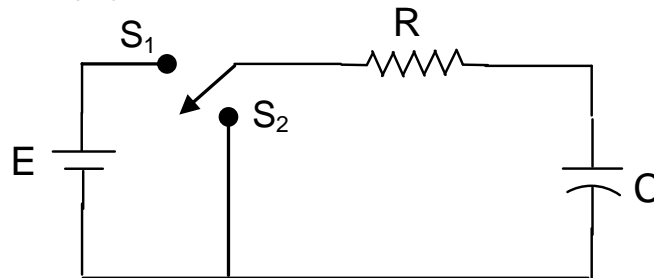
The circuit current and the voltage across the capacitor vary as shown in Fig. below.



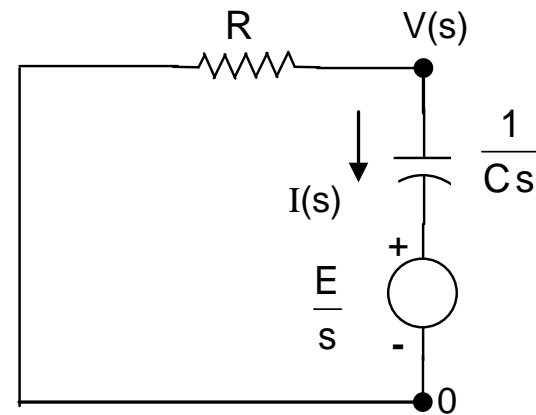
Now, consider the circuit shown in Fig. (a). The switch was in position S_1 for sufficiently long time to establish steady state condition. At time $t = 0$, it is moved to position S_2 .

Before the switch is moved to position S_2 , the capacitor gets charged to voltage E . Since the voltage across the capacitor maintains continuity,

$$v_C(0^+) = v_C(0^-) = E$$



(a)



(b)

The transform circuit for time $t > 0$ is shown in Fig. (b). From this

$$I(s) = - \frac{E/s}{R + \frac{1}{Cs}} = - \frac{EC}{RCs + 1} = - \frac{E/R}{s + \frac{1}{RC}} \quad (7.89)$$

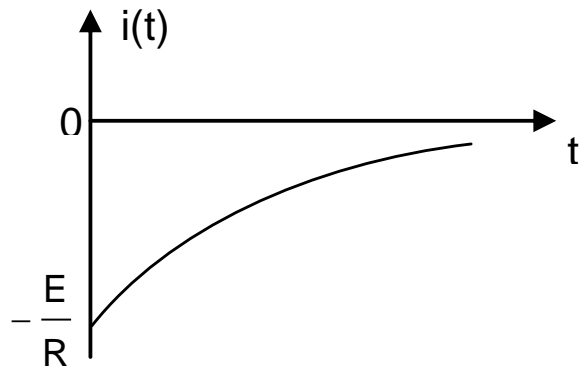
$$\text{Taking inverse LT} \quad i(t) = - \frac{E}{R} e^{-\frac{1}{RC}t} \quad (7.90)$$

It is to be seen that $R I(s) + V_C(s) = 0$

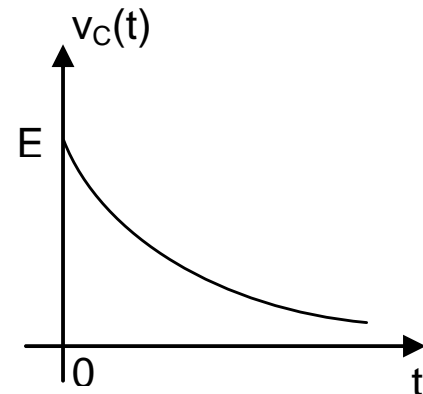
$$\text{Thus } V_C(s) = -R I(s) = \frac{E}{s + \frac{1}{RC}} \quad (7.91)$$

$$\text{Taking inverse LT } v_C(t) = E e^{-\frac{1}{RC}t} \quad (7.92)$$

The wave form of circuit current and the capacitor voltage are shown in Fig. 7.71.



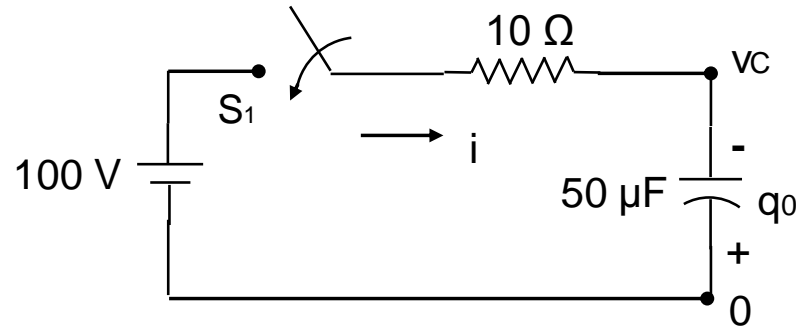
(a)



(b)

Fig. 7.71 Plot of $i(t)$ and $v_C(t)$ as given by Eq. (7.90) and (7.92).

Example 7.28 In the RC circuit shown below, the capacitor has an initial charge $q_0 = 2500 \mu\text{C}$. At time $t = 0$, the switch is closed. Find the circuit current for time $t > 0$.



Solution

$$v_c(0) = - \frac{q_0}{C} = - \frac{2500 \times 10^{-6}}{50 \times 10^{-6}} = -50 \text{ V}$$

Transform circuit for time $t > 0$ is shown in Fig. 7.73.

Referring to Fig. 7.73,

$$I(s) = \frac{\frac{100}{s} + \frac{50}{s}}{10 + \frac{20000}{s}} = \frac{150}{10s + 20000} = \frac{15}{s + 2000}$$

Taking inverse LT, current $i(t) = 15 e^{-2000t} \text{ A}$

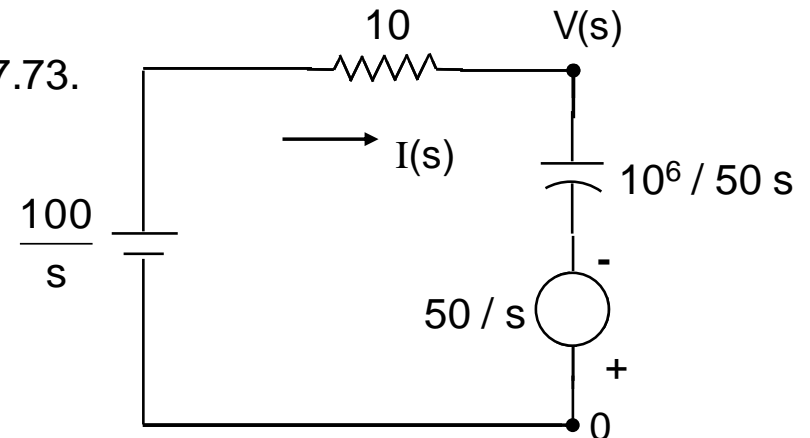
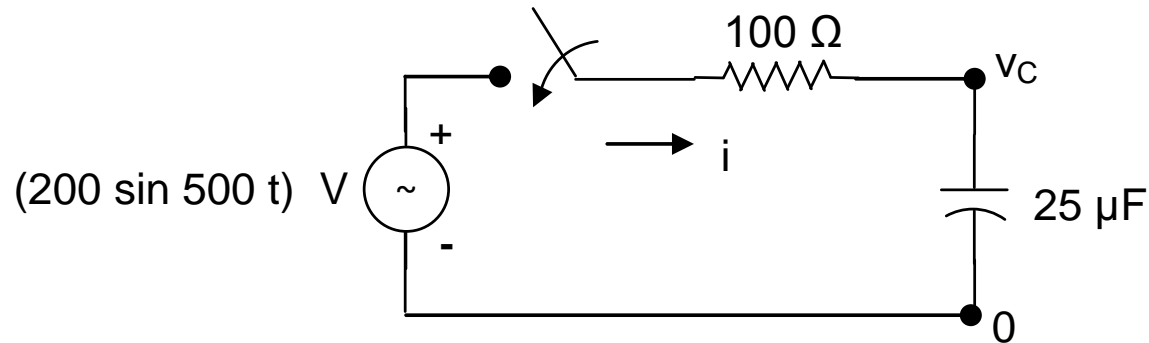


Fig. 7.73 Circuit - Example 7.28.

Example 7.29 For the circuit shown below, find the transient current, assuming that the initial charge on the capacitor as zero, when the switch is closed at time $t = 0$.



Solution $E(s) = \frac{200 \times 500}{s^2 + 250000}; \quad \frac{1}{Cs} = \frac{10^6}{25s}$

$$\text{Therefore, } I(s) = \frac{\frac{10^5}{s^2 + 250000}}{100 + \frac{4 \times 10^4}{s}} = \frac{10^5 s}{(s^2 + 250000)(100s + 4 \times 10^4)}$$

$$= \frac{1000 s}{(s^2 + 250000)(s + 400)} = \frac{K_1 s + K_2}{s^2 + 250000} + \frac{K_3}{s + 400}$$

$$K_3 = \frac{1000 s}{s^2 + 250000} \bigg|_{s = -400} = -0.9756$$

$$\begin{aligned} \text{Further, } 1000 s &= (K_1 s + K_2) (s + 400) - 0.9756 (s^2 + 250000) \\ &= (K_1 - 0.9756) s^2 + (400 K_1 + K_2) s + (400 K_2 - 0.9756 \times 250000) \end{aligned}$$

Comparing the coefficients, in LHS and RHS we have

$$K_1 - 0.9756 = 0 \text{ and } 400 K_1 + K_2 = 1000$$

On solving, $K_1 = 0.9756$; $K_2 = 609.76$

$$\begin{aligned} \text{Thus, } I(s) &= \frac{0.9756 s}{s^2 + 250000} + \frac{609.76}{s^2 + 250000} - \frac{0.9756}{s + 400} \\ &= 0.9756 \frac{s}{s^2 + 500^2} + 1.2195 \frac{500}{s^2 + 500^2} - \frac{0.9756}{s + 400} \end{aligned}$$

Taking inverse LT $i(t) = 0.9756 \cos 500 t + 1.2195 \sin 500 t - 0.9756 e^{-400t} \text{ A}$

Knowing that $\sqrt{(0.9756)^2 + (1.2195)^2} = 1.5617$ and $\tan^{-1} (0.9756 / 1.2195) = 38.66^\circ$

current $i(t) = 1.5617 \sin (500 t + 38.66^\circ) - 0.9756 e^{-400t} \text{ A}$

7.8.3 RLC CIRCUIT

Consider the RLC series circuit shown in Fig. 7.75 (a). Assume that there is no initial charge on the capacitor and there is no initial current through the inductor. The switch is closed at time $t = 0$. Transform circuit for time $t > 0$ is shown in Fig. 7.75 (b).

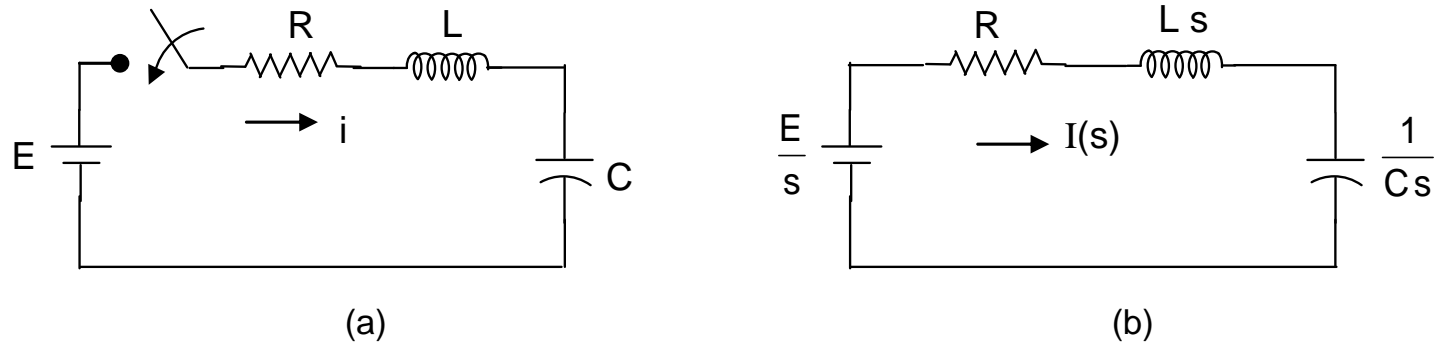


Fig. 7.75 Time domain and s domain - RLC circuit.

Using the transform circuit, expression for the current is obtained as

$$I(s) = \frac{E/s}{R + Ls + \frac{1}{Cs}} = \frac{EC}{RCs + LCs^2 + 1} = \frac{E/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \quad (7.93)$$

The roots of the denominator polynomial are

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = \alpha \pm \beta \quad (7.94)$$

$$\text{where } \alpha = -\frac{R}{2L} \text{ and } \beta = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \quad (7.95)$$

Depending on whether $(\frac{R}{2L})^2 > \frac{1}{LC}$, $(\frac{R}{2L})^2 = \frac{1}{LC}$ or $(\frac{R}{2L})^2 < \frac{1}{LC}$ the discriminant value will be positive, zero or negative and three different cases of solutions are possible.

The value of R , for which the discriminant is zero, is called the critical resistance, R_C .

$$\text{Then } \frac{R_C^2}{4L^2} = \frac{1}{LC};$$

$$\text{Thus } R_C = 2\sqrt{\frac{L}{C}} \quad (7.96)$$

If the circuit resistance $R > R_C$, then $(\frac{R}{2L})^2 > \frac{1}{LC}$.

If the circuit resistance $R < R_C$, then $(\frac{R}{2L})^2 < \frac{1}{LC}$.

Case 1

$$\left(\frac{R}{2L}\right)^2 > \frac{1}{LC} \text{ i.e. } R > R_C \quad (7.97)$$

The two roots s_1 and s_2 are real and distinct. $s_1 = \alpha + \beta$ and $s_2 = \alpha - \beta$ (7.98)

$$\text{Then, } I(s) = \frac{K_1}{s - (\alpha + \beta)} + \frac{K_2}{s - (\alpha - \beta)} \quad (7.99)$$

Taking inverse LT, we get

$$i(t) = K_1 e^{(\alpha + \beta)t} + K_2 e^{(\alpha - \beta)t} = e^{\alpha t} [K_1 e^{\beta t} + K_2 e^{-\beta t}] \quad (7.100)$$

Its plot is shown in Fig. 7.76. In this case the current is said to be over-damped.

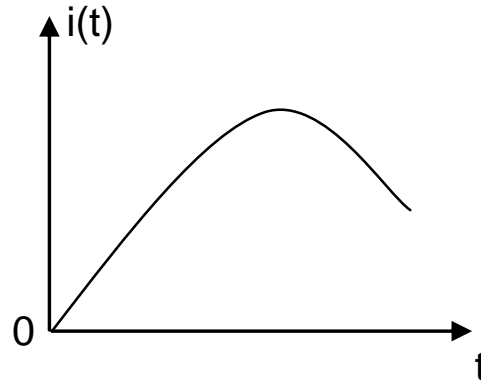


Fig. 7.76 RLC circuit over-damped response.

Case 2

$$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC} \text{ i.e. } R = R_c \quad (7.101)$$

$$\text{Then, } \beta = 0 \text{ and hence the roots are } s_1 = s_2 = \alpha \quad (7.102)$$

$$\text{Thus, } I(s) = \frac{E/L}{(s - \alpha)^2} = \frac{K}{(s - \alpha)^2} \quad (7.103)$$

$$\text{Taking inverse LT, we get } i(t) = K t e^{\alpha t} \quad (7.104)$$

The plot of this current transient is shown in Fig. 7.77. In this case, the current is said to be critically damped.

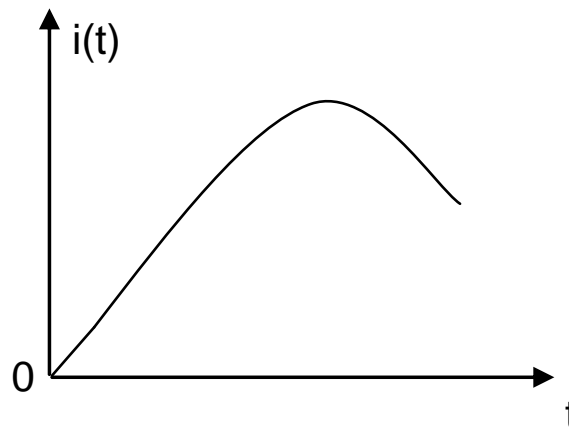


Fig. 7.77 RLC circuit critically-damped response.

Case 3 $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$ i.e. $R < R_c$ (7.105)

For this case, the roots are complex conjugate, $s_1 = \alpha + j\beta$ and $s_2 = \alpha - j\beta$ (7.106)

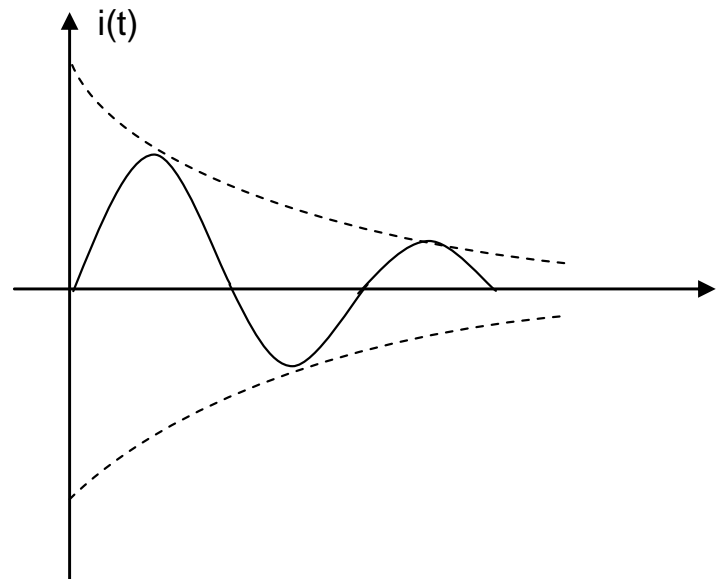
Then, $I(s) = \frac{E/L}{(s - \alpha - j\beta)(s - \alpha + j\beta)} = \frac{E/L}{(s - \alpha)^2 + \beta^2} = \frac{E}{L\beta} \frac{\beta}{(s - \alpha)^2 + \beta^2}$ (7.107)

$= A \frac{\beta}{(s - \alpha)^2 + \beta^2}$ (7.108)

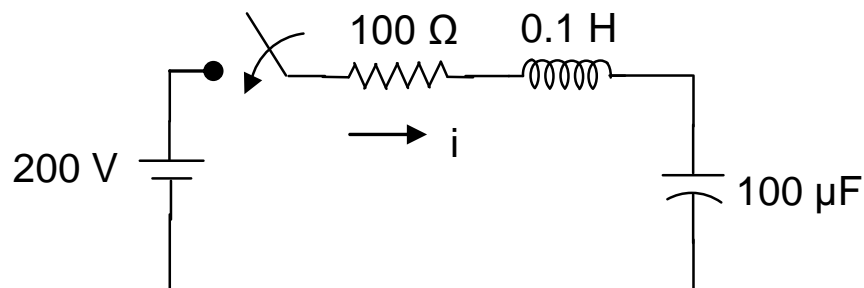
Taking inverse LT, we get $i(t) = A e^{\alpha t} \sin \beta t$ (7.109)

As seen in Equation 7.95, α will be a negative number. Thus, for this **under damped** case, the current is oscillatory and at the same time it decays.

Waveform shown is a
exponentially decaying
sinusoidal wave



Example 7.30 For the RLC circuit shown, find the expression for the transient current when the switch is closed at time $t = 0$. Assume initially relaxed circuit conditions.



Solution The transform circuit is shown in Fig. 7.80.

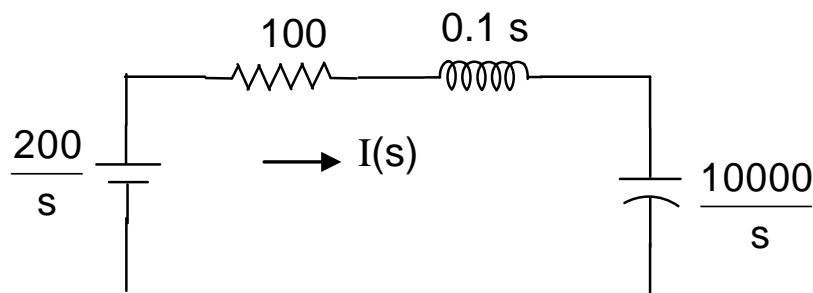


Fig. 7.80 Transform circuit - Example 7.30.

$$\text{Current } I(s) = \frac{200/s}{100 + 0.1s + \frac{10000}{s}} = \frac{200}{0.1s^2 + 100s + 10000} = \frac{2000}{s^2 + 1000s + 100000}$$

$$\text{Current } I(s) = \frac{200/s}{100 + 0.1s + \frac{10000}{s}} = \frac{200}{0.1s^2 + 100s + 10000} = \frac{2000}{s^2 + 1000s + 100000}$$

The roots of the denominator polynomial are

$$s_1, s_2 = \frac{-10^3 \pm \sqrt{10^6 - 0.4 \times 10^6}}{2} = -1127 \text{ and } -887.3$$

$$\text{Therefore, } I(s) = \frac{2000}{(s + 1127)(s + 887.3)} = \frac{K_1}{s + 1127} + \frac{K_2}{s + 887.3}$$

$$K_1 = \left. \frac{2000}{s + 887.3} \right|_{s = -112.7} = 2.582$$

$$K_2 = \left. \frac{2000}{s + 112.7} \right|_{s = -887.3} = -2.582$$

$$\text{Thus, } I(s) = 2.582 \left[\frac{1}{s + 112.7} - \frac{1}{s + 887.3} \right]$$

Taking inverse LT, we get current $i(t) = 2.582 (e^{-112.7t} - e^{-887.3t}) \text{ A}$

This is an example for over-damped.

Example 7.31 Taking the initial conditions as zero, find the transient current in the circuit shown in Fig. 7.81 when the switch is closed at time $t = 0$.

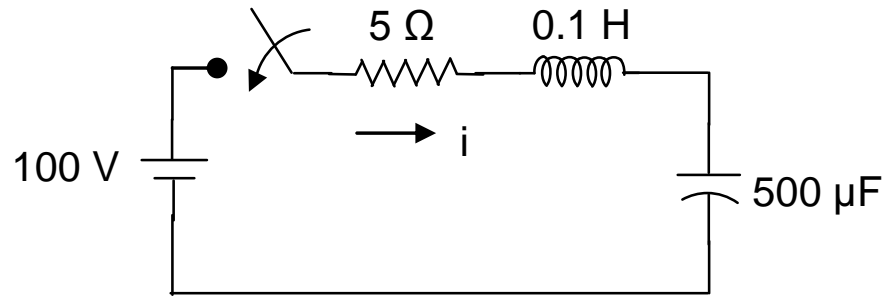


Fig. 7.81 Circuit for Example 7.31.

Solution The transform circuit is shown in Fig. 7.82.

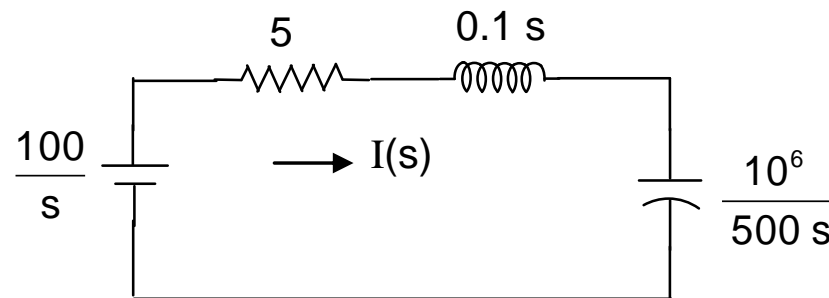


Fig. 7.82 Transform circuit - Example 7.31.

$$\text{Current } I(s) = \frac{100/s}{5 + 0.1s + \frac{10^6}{500s}} = \frac{100}{0.1s^2 + 5s + 2000} = \frac{1000}{s^2 + 50s + 20000}$$

$$\text{Current } I(s) = \frac{100/s}{5 + 0.1s + \frac{10^6}{500s}} = \frac{100}{0.1s^2 + 5s + 2000} = \frac{1000}{s^2 + 50s + 20000}$$

The roots of the denominator polynomial are

$$s_1, s_2 = \frac{-50 \pm \sqrt{2500 - 80000}}{2} = -25 \pm j139.1941$$

It can be seen that

$$s^2 + 50s + 20000 = (s + 25)^2 + (139.1941)^2$$

$$\text{Thus, } I(s) = \frac{1000}{(s + 25)^2 + (139.1941)^2} = 7.1842 \frac{139.1941}{(s + 25)^2 + (139.1941)^2}$$

Taking inverse LT, we get $i(t) = 7.1842 e^{-25t} \sin(139.1941 t) \text{ A}$

This is an example for under-damped.